Discrete Mathematics

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Today's Topics

Conditional Propositions

Logical Equivalence

Converse

Biconditional Propositions

Contrapositive

Proof by Truth Table

Quantifiers

Universal Quantifier

Existential Quantifier

Generalized De Morgan's Laws

Proof by Case Analysis

Logic and Proofs

Conditional Propositions and Logical Equivalence

- Definition
  - If p and q are propositions, the proposition

    if p then q

    is called a conditional proposition and denoted $p \to q$.
  - The proposition p: the hypothesis (or antecedent)
  - The proposition q: the conclusion (or consequent).
- Example
  - If the Mathematics Department gets an additional $40,000, then it will hire one new faculty member.
  - p:
  - q:

Conditional Propositions

- Definition
  - The truth value of the conditional proposition $p \to q$ is defined by the following truth table.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

  - *Note:* $p \to q$ is true when both p and q are true or when p is false.

Example

- Assume that p is true, q is false, and r is true.
- Find the truth value of each proposition below.

  (a) $p \wedge q \to r$

  (b) $p \vee q \to \neg r$

  (c) $p \wedge (q \to r)$

  (d) $p \to (q \to r)$
- A conditional proposition that is true because the hypothesis is false is said to be true by default or vacuously true.

Example

- Restate each proposition below in the form of a conditional proposition.
  - Mary will be a good student if she studies hard.
  - John takes calculus only if he has sophomore, junior, or senior standing.
  - When you sing, my ears hurt.
  - A <u>necessary condition</u> for the Cubs to win the World Series is that they sign a right-handed relief pitcher.
  - A <u>sufficient condition</u> for Maria to visit France is that she goes to the Eiffel Tower.

Converse

- We call the proposition $q \to p$ the converse of the proposition $p \to q$.

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Biconditional Proposition

- Definition
  - If p and q are propositions, the proposition "p if and only if q" is called a biconditional proposition and is denoted $p \leftrightarrow q$. It is sometimes written "p iff q".
  - The truth value of the proposition $p \leftrightarrow q$ is defined by the following truth table.

p	q	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Biconditional Proposition

- An alternative way to state “*p* if and only if *q*” is “p is a necessary and sufficient condition for *q*.”
- Example
  - $1 < 5$ if and only if $2 < 8$.
  - An alternative way to state it is:
    - A necessary and sufficient condition for $1 < 5$ is $2 < 8$.

Logical Equivalence

- Definition
  - Suppose that the propositions P and Q are made up of the propositions $p_1, \ldots, p_n$.
  - We say that P and Q are logically equivalent and write $P \equiv Q$, provided that, given any truth values of $p_1, \ldots, p_n$, either P and Q are both true, or P and Q are both false.

Logical Equivalence

- Definition
  - $\neg p \vee q$ is *logically equivalent* to $p \rightarrow q$

p	q	$\neg p \vee q$	$p \rightarrow q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Logical Equivalence

- Examples
  - Verify the first of De Morgan's laws
    - $\neg(p \vee q) \equiv \neg p \wedge \neg q$
    - $\neg(p \wedge q) \equiv \neg p \vee \neg q$
  - Show that the negation of $p \to q$ is logically equivalent to $p \wedge \neg q$.
  - What is the negation of the proposition "If Jerry receives a scholarship, then he goes to college" in words?
  - Is $p \leftrightarrow q$ logically equivalent to $(p \to q) \wedge (q \to p)$?

Contrapositive

- Definition
  - The *contrapositive* (or *transposition*) of the conditional proposition $p \to q$ is $\neg q \to \neg p$.
- Theorem
  - The conditional proposition and its contrapositive are logically equivalent.
  - Proof.

p	q	$p \to q$	$\neg q \to \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Quantifiers

- Example
  - Let *p*: *n* is an odd integer.
  - Is *p* a proposition?
- Definition
  - Let *P*(*x*) be a statement involving the variable *x* and let *D* be a set.
  - We call *P* a propositional function or predicate (with respect to *D*) if for each *x* in *D*, *P*(*x*) is a proposition.
  - We call *D* the domain of discourse of *P*.

Quantifiers

- Example
  - Let $P(n)$ be the statement

    n is an odd integer,

    and let D be the set of positive integers.
  - Then P is a propositional function with domain of discourse D since for each n in D, $P(n)$ is a proposition.

Quantifiers

- Are any of the following propositional functions?
  - $n^2 + 2n$ is an odd integer
    (domain of discourse = set of positive integers).
  - $x^2 - x - 6 = 0$
    (domain of discourse = set of real numbers).
  - The baseball player hit over .300 in 2003
    (domain of discourse = set of baseball players).
  - The restaurant rated over two stars in *Chicago* magazine
    (domain of discourse = restaurants rated in *Chicago* magazine).

Universal Quantifier

- Definition
  - Let P be a propositional function with domain of discourse D.
  - The statement "for every x, $P(x)$" is said to be a universally quantified statement.
  - The symbol $\forall$ means "for every" in the statement "$\forall x\ P(x)$".
  - The statement $\forall x\ P(x)$ is true if $P(x)$ is true for every x in D.
  - The statement $\forall x\ P(x)$ is false if $P(x)$ is false for at least one x in D.
  - A value x in the domain of discourse that makes $P(x)$ false is called a counterexample to the statement $\forall x\ P(x)$.

Universal Quantifier

- Example
  - Consider the universally quantified statement $\forall x(x^2 \geq 0)$ with domain of discourse the set of real numbers.
  - The statement is true because, *for every* real number x, it is true that the square of x is positive or zero.
- Variables
  - We call the variable x in the propositional function $P(x)$ a free variable. We call the variable x in the universally quantified statement $\forall x\, P(x)$ a bound variable.
  - Note: A statement with free variables is not a proposition, and a statement with no free variables is a proposition.

Universal Quantifier

- Show that the universally quantified statement "for every real number x, if $x > 1$, then $x + 1 > 1$" is true.
  - Proof.
    - Let x be any real number. It is true that for any real number x, either $x \leq 1$ or $x > 1$. If $x \leq 1$, the conditional proposition is vacuously true.
    - Now suppose that $x > 1$. Regardless of the specific value of x, $x + 1 > x$. Since $x + 1 > x$ and $x > 1$, we conclude that $x + 1 > 1$, so the conclusion is true. If $x > 1$, the hypothesis and conclusion are both true hence the conditional proposition is true.
    - We have shown that for every real number x, the proposition "if $x > 1$, then $x + 1 > 1$" is true.
    - Therefore, the universally quantified statement is true.

Existential Quantifier

- Definition
  - Let P be a propositional function with domain of discourse D.
  - The statement "there exists x, $P(x)$" is said to be an existentially quantified statement.
  - The symbol $\exists$ means "there exists," and is called an existential quantifier.
  - The statement $\exists x\ P(x)$ is true if $P(x)$ is true for at least one x in D. The statement $\exists x\ P(x)$ is false if $P(x)$ is false for every x in D.
  - Note: The existentially quantified statement $\exists x\ P(x)$ is false if for every x in the domain of discourse, the proposition $P(x)$ is false.

Existential Quantifier

- Show that the existentially quantified statement

  $$\exists x\ (\ 1/(x^2 + 1) > 1)$$

  is false.
- Proof sketch.
  - We must show that $1/(x^2 + 1) > 1$ is false for every real number x. Since $1/(x^2 + 1) > 1$ is false precisely when $1/(x^2 + 1) \leq 1$ is true, we must show that $1/(x^2 + 1) \leq 1$ is true for every real number x.
  - Let x be any real number. Since $0 \leq x^2$, we obtain $1 \leq x^2 + 1$. If we divide both sides of this last inequality by $x^2 + 1$, we obtain $1/(x^2 + 1) \leq 1$.

Generalized De Morgan's Laws for Logic

- Theorem
  - If P is a propositional function, each pair of propositions in (a) and (b) has the same truth values.

    (a) $\neg(\forall x\, P(x))$; $\exists x\, \neg P(x)$

    (b) $\neg(\exists x\, P(x))$; $\forall x\, \neg P(x)$

  - Proof.
    - Exercise

Summary

- Conditional Propositions
- Logical Equivalence
- Necessary Condition
- Sufficient Condition
- Converse
- Biconditional Propositions
- Contrapositive
- Proof by Truth Table
- Quantifiers
- Universal Quantifier
- Existential Quantifier
- Generalized De Morgan's Laws