Today's Topics
- Conditional Propositions
- Logical Equivalence
- Converse
- Biconditional Propositions
- Contrapositive

Proof by Truth Table
- Quantifiers
- Universal Quantifier
- Existential Quantifier
- Generalized De Morgan's Laws
- Proof by Case Analysis

Logic and Proofs
Conditional Propositions and Logical Equivalence

• Definition
  – If \( p \) and \( q \) are propositions, the proposition
    \[
    \text{if } p \text{ then } q
    \]
  is called a conditional proposition and denoted \( p \rightarrow q \).
  – The proposition \( p \): the hypothesis (or antecedent)
  – The proposition \( q \): the conclusion (or consequent).

• Example
  – If the Mathematics Department gets an additional $40,000,
    then it will hire one new faculty member.
  – \( p \):
  – \( q \):
Conditional Propositions

• Definition
  – The truth value of the conditional proposition \( p \rightarrow q \) is defined by the following truth table.

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– Note: \( p \rightarrow q \) is true when both \( p \) and \( q \) are true or when \( p \) is false.
Example

- Assume that $p$ is true, $q$ is false, and $r$ is true.
- Find the truth value of each proposition below.
  
  (a) $p \land q \rightarrow r$
  
  (b) $p \lor q \rightarrow \neg r$
  
  (c) $p \land (q \rightarrow r)$
  
  (d) $p \rightarrow (q \rightarrow r)$

- A conditional proposition that is true because the hypothesis is false is said to be true by default or vacuously true.
Example

• Restate each proposition below in the form of a conditional proposition.
  – Mary will be a good student if she studies hard.
  – John takes calculus only if he has sophomore, junior, or senior standing.
  – When you sing, my ears hurt.
  – A necessary condition for the Cubs to win the World Series is that they sign a right-handed relief pitcher.
  – A sufficient condition for Maria to visit France is that she goes to the Eiffel Tower.
We call the proposition $q \rightarrow p$ the converse of the proposition $p \rightarrow q$.

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Biconditional Proposition

• Definition
  – If $p$ and $q$ are propositions, the proposition “$p$ if and only if $q$” is called a **biconditional** proposition and is denoted $p \iff q$. It is sometimes written “$p$ iff $q$”.
  – The truth value of the proposition $p \iff q$ is defined by the following truth table.

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An alternative way to state “$p$ if and only if $q$” is “$p$ is a necessary and sufficient condition for $q$.”

Example

- $1 < 5$ if and only if $2 < 8$.
- An alternative way to state it is:
  - A necessary and sufficient condition for $1 < 5$ is $2 < 8$. 
• Definition
  – Suppose that the propositions $P$ and $Q$ are made up of the propositions $p_1, \ldots, p_n$.
  – We say that $P$ and $Q$ are logically equivalent and write $P \equiv Q$, provided that, given any truth values of $p_1, \ldots, p_n$, either $P$ and $Q$ are both true, or $P$ and $Q$ are both false.
**Logical Equivalence**

- **Definition**
  
  \( \neg p \lor q \) is *logically equivalent* to \( p \rightarrow q \)

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Logical Equivalence

Examples

- Verify the first of De Morgan’s laws
  - \( \neg(p \lor q) \equiv \neg p \land \neg q \)
  - \( \neg(p \land q) \equiv \neg p \lor \neg q \)
- Show that the negation of \( p \rightarrow q \) is logically equivalent to \( p \land \neg q \).
- What is the negation of the proposition “If Jerry receives a scholarship, then he goes to college” in words?
- Is \( p \iff q \) logically equivalent to \( (p \rightarrow q) \land (q \rightarrow p) \)?
• **Definition**
  - The *contrapositive* (or *transposition*) of the conditional proposition $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

• **Theorem**
  - The conditional proposition and its contrapositive are logically equivalent.
  
  - **Proof.**

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Quantifiers

• Example
  – Let $p$: $n$ is an odd integer.
  – Is $p$ a proposition?

• Definition
  – Let $P(x)$ be a statement involving the variable $x$ and let $D$ be a set.
  – We call $P$ a propositional function or predicate (with respect to $D$) if for each $x$ in $D$, $P(x)$ is a proposition.
  – We call $D$ the domain of discourse of $P$. 
Quantifiers

• Example
  – Let $P(n)$ be the statement
    
    $n$ is an odd integer,
  
    and let $D$ be the set of positive integers.
  – Then $P$ is a propositional function with
    domain of discourse $D$ since for each $n$ in $D$,
    $P(n)$ is a proposition.
Quantifiers

• Are any of the following propositional functions?
  – $n^2 + 2n$ is an odd integer
    (domain of discourse = set of positive integers).
  – $x^2 - x - 6 = 0$
    (domain of discourse = set of real numbers).
  – The baseball player hit over .300 in 2003
    (domain of discourse = set of baseball players).
  – The restaurant rated over two stars in *Chicago* magazine
    (domain of discourse = restaurants rated in *Chicago* magazine).
Definition

- Let $P$ be a propositional function with domain of discourse $D$.
- The statement “for every $x$, $P(x)$” is said to be a universally quantified statement.
- The symbol $\forall$ means “for every” in the statement “$\forall x \ P(x)$”.
- The statement $\forall x \ P(x)$ is true if $P(x)$ is true for every $x$ in $D$.
- The statement $\forall x \ P(x)$ is false if $P(x)$ is false for at least one $x$ in $D$.
- A value $x$ in the domain of discourse that makes $P(x)$ false is called a counterexample to the statement $\forall x \ P(x)$. 
Universal Quantifier

• Example
  – Consider the universally quantified statement $\forall x(x^2 \geq 0)$ with domain of discourse the set of real numbers.
  – The statement is true because, *for every* real number $x$, it is true that the square of $x$ is positive or zero.

• Variables
  – We call the variable $x$ in the propositional function $P(x)$ a free variable. We call the variable $x$ in the universally quantified statement $\forall x P(x)$ a bound variable.
  – Note: A statement with free variables is not a proposition, and a statement with no free variables is a proposition.
Show that the universally quantified statement “for every real number $x$, if $x > 1$, then $x + 1 > 1$” is true.

Proof.

- Let $x$ be any real number. It is true that for any real number $x$, either $x \leq 1$ or $x > 1$. If $x \leq 1$, the conditional proposition is vacuously true.
- Now suppose that $x > 1$. Regardless of the specific value of $x$, $x + 1 > x$. Since $x + 1 > x$ and $x > 1$, we conclude that $x + 1 > 1$, so the conclusion is true. If $x > 1$, the hypothesis and conclusion are both true hence the conditional proposition is true.
- We have shown that for every real number $x$, the proposition “if $x > 1$, then $x + 1 > 1$” is true.
- Therefore, the universally quantified statement is true.
• **Definition**
  - Let $P$ be a propositional function with domain of discourse $D$.
  - The statement “there exists $x$, $P(x)$” is said to be an **existentially quantified statement**.
  - The symbol $\exists$ means “there exists,” and is called an existential quantifier.
  - The statement $\exists x P(x)$ is true if $P(x)$ is true for at least one $x$ in $D$. The statement $\exists x P(x)$ is false if $P(x)$ is false for every $x$ in $D$.
  - **Note:** The existentially quantified statement $\exists x P(x)$ is false if for every $x$ in the domain of discourse, the proposition $P(x)$ is false.
• Show that the existentially quantified statement
\[ \exists x \ ( \frac{1}{x^2 + 1} > 1 ) \]
is false.

• Proof sketch.
  – We must show that \( \frac{1}{x^2 + 1} > 1 \) is false for every real number \( x \). Since \( \frac{1}{x^2 + 1} > 1 \) is false precisely when \( \frac{1}{x^2 + 1} \leq 1 \) is true, we must show that \( \frac{1}{x^2 + 1} \leq 1 \) is true for every real number \( x \).
  – Let \( x \) be any real number. Since \( 0 \leq x^2 \), we obtain \( 1 \leq x^2 + 1 \). If we divide both sides of this last inequality by \( x^2 + 1 \), we obtain \( \frac{1}{x^2 + 1} \leq 1 \).
Generalized De Morgan’s Laws for Logic

• Theorem
  – If $P$ is a propositional function, each pair of propositions in (a) and (b) has the same truth values.
    
    (a) $\neg(\forall x \ P(x))$; $\exists x \ \neg P(x)$
    
    (b) $\neg(\exists x \ P(x))$; $\forall x \ \neg P(x)$

  – Proof.

• Exercise
Summary

- Conditional Propositions
- Logical Equivalence
- Necessary Condition
- Sufficient Condition
- Converse
- Biconditional Propositions
- Contrapositive
- Proof by Truth Table
- Quantifiers
- Universal Quantifier
- Existential Quantifier
- Generalized De Morgan’s Laws