Discrete Mathematics

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Jong C. Park
Computer Science Division, KAIST

Today's Topics

Mathematical Induction

Strong Form of Induction and the Well-Ordering Property

Logic and Proofs

Mathematical Induction

- Motivating Example
  - Suppose that a sequence of blocks numbered 1, 2, ... sits on an (infinitely) long table and that some blocks are marked with an “X”.
  - Suppose further that

    (a) the first block is marked; and

    (b) for all n, if block n is marked, then block n+1 is also marked.
  - We claim that the statements (a) and (b) imply that every block is marked.

Mathematical Induction

- Another Example
  - Let S_n denote the sum of the first n positive integers:

    $$S_n = 1 + 2 + \ldots + n.$$

  - Suppose that someone claims that

    $$S_n = n(n + 1)/2 \text{ for all } n \geq 1.$$

  - To prove this claim, we show that for all n, if equation n is true, then equation $n + 1$ is also true.
    - We may use a direct proof.
  - We conclude that all of the equations are true.

Mathematical Induction

- Principle of Mathematical Induction
  - Suppose that we have a propositional function $S(n)$ whose domain of discourse is the set of positive integers.
  - Suppose that

    $S(1)$ is true; Basis Step

    for all $n \geq 1$, if $S(n)$ is true, then

    $S(n + 1)$ is true. Inductive Step
  - Then $S(n)$ is true for every positive integer n.

Mathematical Induction

- Definition
  - *n factorial* is defined as

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ n(n-1)(n-2)\cdots 2\cdot 1 & \text{if } n \geq 1. \end{cases}$$

  - That is, if $n \geq 1$, $n!$ is equal to the product of all the integers between 1 and n inclusive.
  - As a special case, 0! is defined to be 1.

Mathematical Induction

- Example
  - Use induction to show that $n! \geq 2^{n-1}$ for all $n \geq 1$.
  - Proof sketch.
    - Basis Step
      - We must show that the inequality holds if $n = 1$.
    - Inductive Step
      - We assume that the inequality holds for n; that is, we assume that $n! \geq 2^{n-1}$ is true.
      - We must then prove that the inequality holds for $n + 1$; that is, we must prove that $(n + 1)! \geq 2^n$ holds.
    - Since the Basis Step and the Inductive Step have been verified, the Principle of Mathematical Induction tells us that the inequality holds for every positive integer n.

Mathematical Induction

- Example
  - If we want to verify that the statements

    $S(n_0), S(n_0 + 1), ...,$

    where $n_0 \neq 1$, are true, we must change the Basis Step to $S(n_0)$ *is true*.
  - The Inductive Step then becomes

    for all $n \geq n_0$, if $S(n)$ is true, then

    $S(n + 1)$ is true.

Mathematical Induction

- Example
  - Geometric Sum
    - Use induction to show that if $r \neq 1$,

      $$a + ar^1 + ar^2 + \ldots + ar^n = a(r^{n+1} - 1)/(r - 1)$$

      for all $n \geq 0$.
    - Proof.
      - exercise
  - Use induction to show that $5^n - 1$ is divisible by 4 for all $n \geq 1$.
    - Proof.
      - exercise

Strong Form of Induction and the Well-Ordering Property

- Strong Form of Mathematical Induction
  - Suppose that we have a propositional function $S(n)$ whose domain of discourse is the set of integers greater than or equal to n_0.
  - Suppose that

    $S(n_0)$ is true;

    for all $n > n_0$, if $S(k)$ is true for all k,

    $n_0 \leq k < n$, then $S(n)$ is true.
  - Then $S(n)$ is true for every integer $n \geq n_0$.
- Show that the two forms of mathematical induction are logically equivalent.

Strong Form of Induction

- Example
  - Use mathematical induction to show that postage of four cents or more can be achieved by using only 2-cent and 5-cent stamps.
    - Proof.
      - exercise
- Example
  - Suppose that the sequence c_1, c_2, ... is defined by the equations $c_1 = 0$, $c_n = c_{\lfloor n/2 \rfloor} + n$ for all $n > 1$.
  - Use strong induction to prove that $c_n < 4n$ for all $n \geq 1$.
    - Proof.
      - exercise

Well-Ordering Property

- The Well-Ordering Property for nonnegative integers states that every nonempty set of nonnegative integers has a least element.
  - Show that this property is equivalent to the two forms of induction.

Today's Topics

Sets

The Language of Mathematics

Sets

- A set is a collection of objects (elements, members).
  - Examples
    - A = {1, 2, 3, 4} = {1, 3, 4, 2} = {1, 2, 2, 3, 4}
    - B = {x | x is a positive, even integer}
- Cardinality
  - If X is a finite set, we let $|X|$ = number of elements in X.
- Membership
  - If x is in the set X, we write $x \in X$, and if x is not in X, we write $x \notin X$.

Sets

- Empty Set
  - The set with no elements is called the empty (null, void) set and is denoted $\varnothing$. Thus $\varnothing = \{ \}$.
- Equality
  - Two sets X and Y are equal, notated as $X = Y$, if X and Y have the same elements. In symbols, $X = Y$ iff $\forall x((x \in X \rightarrow x \in Y) \wedge (x \in Y \rightarrow x \in X))$.
- Example
  - Prove that if $A = \{x \mid x^2 + x - 6 = 0\}$ and $B = \{2, -3\}$, then $A = B$.

Sets

- Subset
  - Suppose that X and Y are sets. If every element of X is an element of Y, we say that X is a subset of Y, written as $X \subseteq Y$. In symbols, X is a subset of Y if $\forall x(x \in X \rightarrow x \in Y)$.
- Examples
  - If $C = \{1,3\}$ and $A = \{1,2,3,4\}$, then $C \subseteq A$.
  - Show that $X \subseteq Y$, where $X = \{x \mid x^2 + x - 2 = 0\}$, Y = set of integers, and the domain of discourse is the set of real numbers.
  - Show that if $X = \{x \mid 3x^2 - x - 2 = 0\}$ and Y = set of integers, X is not a subset of Y.

Sets

- Proper Subset
  - If X is a subset of Y and X does not equal Y, we say that X is a proper subset of Y and write $X \subset Y$.
- Power Set
  - The set of all subsets (proper or not) of a set X, denoted $\wp(X)$, is called the power set of X.
  - Example
    - If $A = \{a,b,c\}$, the members of $\wp(A)$ are $\varnothing$, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}. All but {a,b,c} are proper subsets of A.
    - For this example, $|A| = 3$, $|\wp(A)| = 2^3 = 8$.
  - Show that if $|X| = n$, then $|\wp(X)| = 2^n$.
    - Proof.
      - By induction on n.

Sets

- Union
  - Given two sets X and Y, the set $X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$ is called the union of X and Y.
- Intersection
  - The set $X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$ is called the intersection of X and Y.
- Difference
  - The set $X - Y = \{x \mid x \in X \text{ and } x \notin Y\}$ is called the difference (or relative complement).
- Disjoint
  - Sets X and Y are disjoint if $X \cap Y = \varnothing$.
  - A collection of sets S is said to be pairwise disjoint if whenever X and Y are distinct sets in S, X and Y are disjoint.

Sets

- Universe
  - Sometimes we are dealing with sets, all of which are subsets of a set U. This set U is called a universal set or a universe. The set U must be explicitly given or inferred from the context.
  - Given a universal set U and a subset X of U, the set $U – X$ is called the complement of X and is written X^C.
- Venn diagrams
  - Venn diagrams provide pictorial views of sets. In a Venn diagram, a rectangle depicts a universal set. Subsets of the universal set are drawn as circles. The inside of a circle represents the members of that set.

Summary

- Mathematical Induction
- Strong Form of Induction and the Well-Ordering Property
- Sets