Discrete Mathematics

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Jong C. Park
Computer Science Division, KAIST

Today's Topics

Sets

Functions

The Language of Mathematics

Sets

- Theorem 2.1.12

  Let U be a universal set and let A, B, and C be subsets of U.

  Verify the following properties.

  (a) Associative laws
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$

  (b) Commutative laws
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$

  (c) Distributive laws
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Sets

(d) Identity laws

- $A \cap U = A$
- $A \cup \varnothing = A$

(e) Complement laws

- $A \cup A^c = U$
- $A \cap A^c = \varnothing$

(f) Idempotent laws

- $A \cup A = A$
- $A \cap A = A$

(g) Bound laws

- $A \cup U = U$
- $A \cap \varnothing = \varnothing$

(h) Absorption laws

- $A \cup (A \cap B) = A$
- $A \cap (A \cup B) = A$

(i) Involution law

- $(A^c)^c = A$

(j) 0/1 laws

- $\varnothing^c = U$
- $U^c = \varnothing$

(k) De Morgan's laws for sets

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

Sets

- $\cup S$
  - We define the union of an arbitrary family S of sets to be those elements x belonging to at least one set X in S.
    - Formally, $\cup S = \{x \mid x \in X \text{ for some } X \in S\}$.
  - Similarly, we define the intersection of an arbitrary family S of sets to be those elements x belonging to every set X in S.
    - Formally, $\cap S = \{x \mid x \in X \text{ for all } X \in S\}$.
  - If $S = \{A_1, A_2, \ldots, A_n\}$, we write

$$\cup S = \cup^{n}_{i=1} A_i, \cap S = \cap^{n}_{i=1} A_i.$$

  - If $S = \{A_1, A_2, \ldots\}$, we write

$$\cup S = \cup^{\infty}_{i=1} A_i, \cap S = \cap^{\infty}_{i=1} A_i.$$

Sets

- Example
  - For $i \geq 1$, define $A_i = \{i, i + 1, ...\}$ and $S = \{A_1, A_2, ...\}$.
  - Then $\cup^{\infty}_{i=1} A_i = \cup S = \{1, 2, ...\}$, $\cap^{\infty}_{i=1} A_i = \cap S = \varnothing$.
- Partition
  - A collection S of nonempty subsets of X is said to be a partition of the set X if every element in X belongs to exactly one member of S.
    - Note that if S is a partition of X, S is pairwise disjoint and $\cup S = X$.
  - Example
    - Since each element of $X = \{1,2,3,4,5,6,7,8\}$ is in exactly one member of $S = \{\{1,4,5\},\{2,6\},\{3\},\{7,8\}\}$, S is a partition of X.

Sets

- Cartesian product
  - An ordered pair of elements, written (a,b), is considered distinct from the ordered pair (b,a), unless $a = b$.
  - If X and Y are sets, we let $X \times Y$ denote the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. We call $X \times Y$ the Cartesian product of X and Y.
- Example
  - If $X = \{1,2,3\}$ and $Y = \{a,b\}$, then
    - $X \times Y = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$
    - $Y \times X = \{(a,1), (b,1), (a,2), (b,2), (a,3), (b,3)\}$
    - $X \times X = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$
    - $Y \times Y = \{(a,a), (a,b), (b,a), (b,b)\}$

Sets

- n-tuple
  - An n-tuple, written $(a_1, a_2, ..., a_n)$, takes order into account; that is, $(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n)$ precisely when $a_1 = b_1$, $a_1 = b_1$, ..., $a_1 = b_1$,
  - The Cartesian product of sets $X_1, X_2, ..., X_n$ is defined to be the set of all n-tuples $(x_1, x_2, ..., x_n)$ where $x_i \in X_i$ for $i = 1, ..., n$; it is denoted $X_1 \times X_2 \times ... \times X_n$.
- Examples
  - Show that for arbitrary finite sets X and Y, $|X \times Y| = |X|\cdot|Y|$.
  - Show that $|X_1 \times X_2 \times ... \times X_n| = |X_1|\cdot|X_2|\cdots|X_n|$.
    - Proof.
      - Use induction on the number n of sets.

Functions

- Definition
  - Let X and Y be sets. A function f from X to Y is a subset of the Cartesian product $X \times Y$ having the property that for each $x \in X$, there is exactly one $y \in X$ with $(x,y) \in f$.
  - We sometimes denote a function f from X to Y as $f: X \to Y$.
  - The set X is called the domain of f.
  - The set $\{y \mid (x,y) \in f\}$ (which is a subset of Y) is called the range of f.

Functions

- Are the following sets functions?
  - The set $f = \{(1,a), (2,b), (3,a)\}$, with the domain $X = \{1,2,3\}$ and the range $Y = \{a,b,c\}$.
  - The set $\{(1,a), (2,a), (3,b)\}$, with the domain $X = \{1,2,3,4\}$ and the range $Y = \{a,b,c\}$.
  - The set $\{(1,a), (2,b), (3,c), (1,b)\}$, with the domain $X = \{1,2,3\}$ and the range $Y = \{a,b,c\}$.
- Visualization of a function
  - Use an arrow diagram
  - Use the graph of a function

Modulus Operator

- Modulus operator
  - If *x* is an integer and *y* is a positive integer, we define *x* mod *y* to be the remainder when *x* is divided by y.
- Examples
  - We have 6 mod 2 = 0, 5 mod 1 = 0, 8 mod 12 = 8.
  - What day of the week will it be 365 days from Wednesday?
    - 365 mod 7 = 1

Modulus Operator

- Example
  - International Standard Book Numbers (ISBN)
    - An ISBN is a code of 10 characters separated by dashes, such as 0-8065-0959-7.
    - An ISBN consists of four parts: a group code, a publisher code, a code that uniquely identifies the book among those published by the particular publisher, and a check character.
    - The check character is *s* mod 11, where *s* is the sum of the first digit plus two times the second digit plus three times the third digit, ..., plus nine times the ninth digit. If this value is 10, the check character is X.
    - $s = 0 + 2 \cdot 8 + 3 \cdot 0 + 4 \cdot 6 + 5 \cdot 5 + 6 \cdot 0 + 7 \cdot 9 + 8 \cdot 5 + 9 \cdot 9 = 249$
    - $249 \bmod 11 = 7$.

Floor and Ceiling

- Definition
  - The floor of x, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x.
  - The ceiling of x, denoted $\lceil x \rceil$, is the least integer greater than or equal to x.
- Examples
  - $\lfloor 8.3 \rfloor = 8$, $\lceil 9.1 \rceil = 10$, $\lfloor -8.7 \rfloor = -9$, $\lceil -11.3 \rceil = -11$, $\lceil 6 \rceil = 6$, $\lfloor -8 \rfloor = -8$.
  - cf. the graphs of the floor and ceiling functions, where a bracket, [or], indicates that the point is to be included in the graph; a parenthesis, (or), indicates that the point is to be excluded from the graph.

Floor and Ceiling

- Example
  - In 2005, the U.S. first-class postage rate for up to 13 ounces was 37 cents for the first ounce or fraction thereof and 23 cents for each additional ounce or fraction thereof.
  - Define the postage $P(w)$ as a function of weight w.
    - $P(w) = 37 + 23\lceil w - 1 \rceil$, $13 \geq w > 0$.

Properties

- Definition
  - A function f from X to Y is said to be one-to-one (or injective) if for each $y \in Y$, there is at most one $x \in X$ with $f(x) = y$.
  - When a function f is one-to-one, for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
  - In symbols, $\forall x_1 \forall x_2((\mathrm{f}(x_1) = \mathrm{f}(x_2)) \rightarrow (x_1 = x_2))$.
- Are the following functions one-to-one?
  - $f = \{(1,b), (3,a), (2,c)\}$ from $X = \{1,2,3\}$ to $Y = \{a,b,c,d\}$
  - $f = \{(1,a), (2,b), (3,a)\}$ from $X = \{1,2,3\}$ to $Y = \{a,b,c,d\}$

Properties

- Further examples
  - Prove that the function $f(n) = 2n + 1$ from the set of positive integers to the set of positive integers is one-to-one.
    - Proof.
      - Use a direct proof.
  - Prove that the function $f(n) = 2^n - n^2$ from the set of positive integers to the set of integers is not one-to-one.

Properties

- Definition
  - If f is a function from X to Y and the range of f is Y, f is said to be onto Y (or an onto function or a surjective function).
  - When a function f is onto, for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.
  - In symbols, $\forall y \in Y\ \exists x \in X\ (f(x) = y)$.
- Are the following functions onto?
  - $f = \{(1,a), (2,c), (3,b)\}$ from $X = \{1,2,3\}$ to $Y = \{a,b,c\}$
  - $f = \{(1,b), (3,a), (2,c)\}$ from $X = \{1,2,3\}$ to $Y = \{a,b,c,d\}$

Properties

- Further examples
  - Prove that the function $f(x) = 1/x^2$ from the set X of nonzero real numbers to the set Y of positive real numbers is onto Y.
  - Prove that the function $f(n) = 2n - 1$ from the set X of positive integers to the set Y of positive integers is not onto Y.

Properties

- Definition
  - A function that is both one-to-one and onto is called a bijection.
- Examples
  - $f = \{(1,a), (2,c), (3,b)\}$ from $X = \{1,2,3\}$ to $Y = \{a,b,c\}$
  - Show that $\{(y,x) \mid (x,y) \in f\}$ is one-to-one, onto function from Y to X, if f is a one-to-one, onto function from X to Y.
    - This new function, denoted f^{-1}, is called f inverse.
    - For the function $f = \{(1,a), (2,c), (3,b)\}$, we have $f^{-1} = \{(a,1), (c,2), (b,3)\}$.

Summary

- Sets
  - Union
  - Intersection
  - Partition
  - Cartesian Product
  - *n*-tuple
- Functions
  - Domain
  - Range
  - Modulus Operator
  - Floor
  - Ceiling
  - Injective Function (One-to-One)
  - Surjective Function (Onto)
  - Bijective Function