Discrete Mathematics

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Today's Topics

Functions (continued)

Sequences and Strings

The Language of Mathematics

Composition

- Definition
  - Let g be a function from X to Y and let f be a function from Y to Z.
  - The composition of f with g, denoted $f \circ g$, is the function $(f \circ g)(x) = f(g(x))$ from X to Z.
- Example
  - Given $g = \{(1,a), (2,a), (3,c)\}$, a function from $X = \{1,2,3\}$ to $Y = \{a,b,c\}$, and $f = \{(a,y), (b,x), (c,z)\}$, a function from Y to $Z = \{x,y,z\}$, the composition function from X to Z is the function $f \circ g = \{(1,y), (2,y), (3,z)\}$.

Arity

- Definition
  - A function from $X \times X$ to X is called a binary operator on X.
  - A function from X to X is called a unary operator on X.
- Examples
  - Let $X = \{1,2,...\}$. If we define $f(x,y) = x + y$, where $x,y \in X$, then f is a binary operator on X.
  - Let U be a universal set. If we define $f(X) = X^C$, where $X \in \wp(U)$, then f is a unary operator on $\wp(U)$.

Sequences and Strings

- Definition
  - A sequence is a special type of function in which the domain consists of a set of consecutive integers.
  - The *n*th term is typically denoted C_n.
  - We call *n* the index of the sequence.
- Examples
  - sequence *s*: 2, 4, 6, ..., 2*n*, ...
  - sequence *t*: *a*, *a*, *b*, *a*, *b*

Sequences

- Definition
  - If the domain of the sequence is infinite, we say that the sequence is infinite. Otherwise, we say that the sequence is finite.
  - When we want to explicitly state the initial index *k* of an infinite sequence *s*, we can write $\{s_n\}^{\infty}_{n=0}$.
  - A finite sequence *x* indexed from *i* to *j* can be denoted $\{x_n\}^{j}_{n=i}$.
- Example
  - A sequence *t* whose domain is {-1, 0, 1, 2, 3} can be denoted $\{t_n\}^{3}_{n=-1}$.

Sequences

- Definition
  - A sequence s is increasing if $s_n < s_{n+1}$ for all n for which n and $n + 1$ are in the domain of the sequence.
  - A sequence s is decreasing if $s_n > s_{n+1}$ for all n for which n and $n + 1$ are in the domain of the sequence.
  - A sequence s is nondecreasing if $s_n \leq s_{n+1}$ for all n for which n and $n + 1$ are in the domain of the sequence.
  - A sequence s is nonincreasing if $s_n \geq s_{n+1}$ for all n for which n and $n + 1$ are in the domain of the sequence.

Sequences

- Identify the type of the following sequences.
  - 2, 5, 13, 104, 300
  - $a_i = 1/i$, $i \geq 1$
  - 100, 90, 90, 74, 74, 74, 30
  - 100

Sequences

- Definition
  - Let $\{s_n\}$ be a sequence defined for $n = m$, $m+1$, ..., and let n_1, n_2, ... be an increasing sequence whose values are in the set $\{m, m+1, ...\}$. We call the sequence $\{s_{n_k}\}$ a subsequence of $\{s_n\}$.
- Examples
  - The sequence b, c is a subsequence of the sequence $t_1 = a$, $t_2 = a$, $t_3 = b$, $t_4 = c$, $t_5 = q$.
  - The sequence 2, 4, 8, 16, ..., $2k$, ... is a subsequence of the sequence 2, 4, 6, 8, 10, 12, 14, 16, ..., $2n$, ...

Sequences

- Definition
  - If $\{a_i\}^n_{i=m}$ is a sequence, we define
    $\Sigma^n_{i=m} a_i = a_m + a_{m+1} + \cdots + a_n$,
    $\Pi^n_{i=m} a_i = a_m \cdot a_{m+1} \cdots a_n$.
  - The formalism $\Sigma^n_{i=m} a_i$ is called the sum (or sigma) notation and $\Pi^n_{i=m} a_i$ is called the product notation.
  - i is called the index, m is called the lower limit, and n is called the upper limit.
- Example
  - Let a be the sequence defined by $a_n = 2n$, $n \geq 1$.
  - Compute $\Sigma^3_{i=1} a_i$ and $\Pi^3_{i=1} a_i$.

Strings

- Definition
  - A string over X, where X is a finite set, is a finite sequence of elements from X.
  - The string with no elements is called the null string, denoted λ.
  - We let X^* denote the set of all strings over X, including the null string, and we let X^+ denote the set of all nonnull strings over X.
- Examples
  - Let $X = \{a,b,c\}$. If we let $\beta_1 = b$, $\beta_2 = a$, $\beta_3 = a$, $\beta_4 = c$, we obtain a string over X. This string is written *baac*.
  - Repetitions in a string can be specified by superscripts.
    - *bbaaac* may be written b^2a^3c.

Strings

- Definition
  - The length of a string α is the number of elements in α, denoted $|\alpha|$.
  - If α and β are two strings, the string consisting of α followed by β, written $\alpha\beta$, is called the concatenation of α and β.
- Examples
  - If $\alpha = aabab$ and $\beta = a^3b^4a^{32}$, then $|\alpha| = 5$ and $|\beta| = 39$.
  - Given $\gamma = aab$ and $\theta = cabd$, compute $\gamma\theta$, $\theta\gamma$, $\gamma\lambda$, $\lambda\gamma$.

Strings

- Definition
  - A string β is a substring of the string α if there are strings γ and δ with $\alpha = \gamma\beta\delta$.
- Example
  - The string β = *add* is a substring of the string α = *aaaddad*.

Today's Topics

Relations

Relations

Relations

- Definition
  - A (binary) relation R from a set X to a set Y is a subset of the Cartesian product $X \times Y$.
  - If $(x,y) \in R$, we write $x\ R\ y$ and say that x is related to y.
  - If $X = Y$, we call R a (binary) relation on X.
  - The set $\{x \in X \mid (x,y) \in R \text{ for some } y \in Y\}$ is called the domain of R.
  - The set $\{y \in Y \mid (x,y) \in R \text{ for some } x \in X\}$ is called the range of R.

Relations

- Note
  - A function is a special type of relation. A function f from X to Y is a relation from X to Y having the properties:
    - (a) The domain of f is equal to X.
    - (b) For each $x \in X$, there is exactly one $y \in Y$ such that $(x,y) \in f$.
- Example
  - For X = {Bill, Mary, Beth, Dave} and Y = {CompSci, Math, Art, History}, we may have a relation R = {(Bill, CompSci), (Mary,Math), (Bill,Art), (Beth,History), (Beth,CompSci), (Dave,Math)}.

Relations

- Example
  - Let R be the relation on $X = \{1,2,3,4\}$ defined by $(x,y) \in R$ if $x \leq y$, $x, y \in X$.
  - Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$.
  - The domain and range of R are both equal to X.
- Note
  - An informative way to picture a relation on a set is to draw its digraph, a notion to be defined later.

Properties

- Definition
  - A relation R on a set X is called reflexive if $(x,x) \in R$ for every $x \in X$.
- Are the following relations reflexive?
  - the relation R on $X = \{1,2,3,4\}$ defined by $(x,y) \in R$ if $x \leq y$, where $x, y \in X$
  - the relation $R = \{(a,a), (b,c), (c,b), (d,d)\}$ on $X = \{a,b,c,d\}$

Properties

- Definition
  - A relation R on a set X is called symmetric if for all x, $y \in X$, if $(x,y) \in R$, then $(y,x) \in R$.
- Are the following relations symmetric?
  - the relation $R = \{(a,a), (b,c), (c,b), (d,d)\}$ on $X = \{a,b,c,d\}$
  - the relation R on $X = \{1,2,3,4\}$ defined by $(x,y) \in R$ if $x \leq y$, $x,y \in X$

Properties

- Definition
  - A relation R on a set X is called antisymmetric if for all x, $y \in X$, if $(x,y) \in R$ and $x \neq y$, then $(y,x) \notin R$.
- Are the following relations antisymmetric?
  - the relation R on $X = \{1,2,3,4\}$ defined by $(x,y) \in R$ if $x \leq y$, $x,y \in X$
  - the relation $R = \{(a,a), (b,c), (c,b), (d,d)\}$ on $X = \{a,b,c,d\}$
  - the relation $R = \{(a,a), (b,b), (c,c)\}$

Properties

- Definition
  - A relation R on a set X is called transitive if for all $x,y,z \in X$, if (x,y) and $(y,z) \in R$, then $(x,z) \in R$.
- Are the following relations transitive?
  - the relation R on $X = \{1,2,3,4\}$ defined by $(x,y) \in R$ if $x \leq y$, $x,y \in X$
  - the relation $R = \{(a,a), (b,c), (c,b), (d,d)\}$ on $X = \{a,b,c,d\}$

Properties

- Definition
  - A relation R on a set X is called a partial order if R is reflexive, antisymmetric, and transitive.
- Example
  - the relation R defined on the positive integer by $(x,y) \in R$ if x divides y
- Note
  - Suppose that R is a partial order on a set X. If $x,y \in X$ and either (x,y) or (y,x) is in R, we say that x and y are comparable. If $x, y \in X$ and both (x,y) and (y,x) are not in R, we say that x and y are incomparable. If every pair of elements in X is comparable, we call R a total order (on X).

Properties

- Definition
  - Let R be a relation from X to Y. The inverse of R, denoted R^{-1}, is the relation from Y to X defined by $R^{-1} = \{(y,x) \mid (x,y) \in R\}$.
- Example
  - If we define a relation R from $X = \{2,3,4\}$ to $Y = \{3,4,5,6,7\}$ by

    $$(x,y) \in R \text{ if } x \text{ divides } y,$$

    we obtain $R = \{(2,4), (2,6), (3,3), (3,6), (4,4)\}$.

    What is the inverse of this relation R?

Composition

- Definition
  - Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The composition of R_1 and R_2, denoted $R_2 \circ R_1$, is the relation from X to Z defined by $R_2 \circ R_1 = \{(x,z) \mid (x,y) \in R_1$ and $(y,z) \in R_2$ for some $y \in Y\}$.
- Example
  - What is the composition of the relations $R_1 = \{(1,2), (1,6), (2,4), (3,4), (3,6), (3,8)\}$ and $R_2 = \{(2,u), (4,s), (4,t), (6,t), (8,u)\}$, or $R_2 \circ R_1$?

Summary

- Functions
- Sequences and Strings
- Relations