Discrete Mathematics

CS204: Spring, 2008

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Today's Topics

Equivalence Relations

Matrices of Relations

Relations

Equivalence Relations

- Theorem
  - Let $\mathcal{S}$ be a partition of a set X. Define $x\ R\ y$ to mean that for some set S in $\mathcal{S}$, both x and y belong to S. Then R is reflexive, symmetric, and transitive.
  - Proof.
    - Let $x \in X$. By the definition of partition, x belongs to some member S of $\mathcal{S}$. Thus $x\ R\ x$ and R is reflexive.
    - Suppose that $x\ R\ y$. Then both x and y belong to some set $S \in \mathcal{S}$. Since both y and x belong to S, $y\ R\ x$ and R is symmetric.
    - Finally, suppose that $x\ R\ y$ and $y\ R\ z$. Then both x and y belong to some set $S \in \mathcal{S}$ and both y and z belong to some set $T \in \mathcal{S}$. Since y belongs to exactly one member of $\mathcal{S}$, we must have $S = T$. Therefore, both x and z belong to S and $x\ R\ z$. We have shown that R is transitive.

Equivalence Relations

- Example
  - Consider the partition $S = \{\{1,3,5\}, \{2,6\}, \{4\}\}$ of $X = \{1,2,3,4,5,6\}$.
  - What is the relation R on X as given by the preceding theorem?

Equivalence Relations

- Definition
  - A relation that is reflexive, symmetric, and transitive on a set X is called an equivalence relation on X.
- Are any of the following relations equivalence relations?
  - the relation $R = \{(1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5)\}$
  - the relation R on $X = \{1,2,3,4\}$ defined by $(x,y) \in R$ if $x \leq y$, $x, y \in X$
  - the relation $R = \{(a,a), (b,c), (c,b), (d,d)\}$ on $X = \{a,b,c,d\}$

Equivalence Relations

- Theorem
  - Let R be an equivalence relation on a set X. For each $a \in X$, let $[a] = \{x \in X \mid x \, R \, a\}$. (In words, $[a]$ is the set of all elements in X that are related to a.) Then $\mathcal{S} = \{[a] \mid a \in X\}$ is a partition of X.
  - Proof.
    - We must show that every element in X belongs to exactly one member of $\mathcal{S}$. Let $a \in X$. Since $a \, R \, a$, $a \in [a]$. Thus every element in X belongs to at least one member of $\mathcal{S}$.
    - It remains to show that every element in X belongs to exactly one member of $\mathcal{S}$; that is,

      if $x \in X$ and $x \in [a] \cap [b]$, then $[a] = [b]$.

Equivalence Relations

– Proof (continued).

- We first show that for all c, $d \in X$, if $c\ R\ d$, then $[c] = [d]$. Suppose that $c\ R\ d$. Let $x \in [c]$. Then $x\ R\ c$. Since $c\ R\ d$ and R is transitive, $x\ R\ d$. Therefore, $x \in [d]$ and $[c] \subseteq [d]$. The argument that $[d] \subseteq [c]$ is the same as that just given, but with the roles of c and d interchanged. Thus $[c] = [d]$.
- We now prove the following claim.

  if $x \in X$ and $x \in [a] \cap [b]$, then $[a] = [b]$.

  Assume that $x \in X$ and $x \in [a] \cap [b]$. Then $x\ R\ a$ and $x\ R\ b$. Our preceding result shows that $[x] = [a]$ and $[x] = [b]$. Thus $[a] = [b]$.

Equivalence Relations

- Definition
  - Let R be an equivalence relation on a set X. The sets $[a]$ as defined in the preceding theorem are called the equivalence classes of X given by the relation R.
- Examples
  - Given the equivalence relation $R = \{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5), (2,2), (2,6), (6,2), (6,6), (4,4)\}$ on $X = \{1,2,3,4,5,6\}$, determine the equivalence classes [1], [2], [3], and [4].
  - What are the equivalence classes for the equivalence relation $R = \{(1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5)\}$?

Equivalence Relations

- Theorem
  - Let R be an equivalence relation on a finite set X. If each equivalence class has r elements, there are $|X|/r$ equivalence classes.
  - Proof.
    - Let $X_1, X_2, ..., X_k$ denote the distinct equivalence classes.
    - Since these sets partition X, $|X| = |X_1| + |X_2| + \cdots + |X_k| = r + r + \cdots + r = kr$ and the conclusion follows.

Matrices of Relations

- Example
  - The matrix of the relation $R = \{(1,b), (1,d), (2,c), (3,c), (3,b), (4,a)\}$ from $X = \{1,2,3,4\}$ to $Y = \{a,b,c,d\}$ relative to the orderings 1, 2, 3, 4 and a, b, c, d is

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 \\ 4 & 1 & 0 & 0 & 0 \end{array}$$

Matrices of Relations

- Example
  - The matrix of the relation $R = \{(1,b), (1,d), (2,c), (3,c), (3,b), (4,a)\}$ from $X = \{1,2,3,4\}$ to $Y = \{a,b,c,d\}$ relative to the orderings 2, 3, 4, 1 and d, b, a, c is

$$\begin{array}{c|cccc} & d & b & a & c \\ \hline 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}$$

Matrices of Relations

- Example
  - The matrix of the relation R from {2,3,4} to {5,6,7,8}, relative to the orderings 2, 3, 4 and 5, 6, 7, 8, defined by

$$x \; R \; y \text{ if } x \text{ divides } y$$

is

	5	6	7	8
2	0	1	0	1
3	0	1	0	0
4	0	0	0	1

Matrices of Relations

- Example
  - The matrix of the relation $R = \{(a,a), (b,b), (c,c), (d,d), (b,c), (c,b)\}$ on $\{a,b,c,d\}$, relative to the ordering a, b, c, d, is

$$
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 1 & 0 & 0 & 0 \\
b & 0 & 1 & 1 & 0 \\
c & 0 & 1 & 1 & 0 \\
d & 0 & 0 & 0 & 1
\end{array}
$$

Matrices of Relations

- How to determine whether a relation R on a set X is
  - reflexive?
    - if and only if A has 1's on the main diagonal
    - if and only if $(x,x) \in R$ for all $x \in X$
  - symmetric?
    - if and only if A is symmetric about the main diagonal
    - if and only if for all i and j, the ijth entry of A is equal to the jith entry of A
  - antisymmetric?
    - whenever the ijth entry is 1, $i \neq j$, the jith entry is not 1

Matrices of Relations

- Example
  - Let R_1 be the relation from $X = \{1,2,3\}$ to $Y = \{a,b\}$ defined by $R_1 = \{(1,a), (2,b), (3,a), (3,b)\}$, and let R_2 be the relation from Y to $Z = \{x,y,z\}$ defined by $R_2 = \{(a,x), (a,y), (b,y), (b,z)\}$.
  - The matrix of R_1 relative to the orderings 1, 2, 3 and a, b is

$$A_1 = \begin{array}{c|cc} & a & b \\ \hline 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{array}$$

Matrices of Relations

– and the matrix of R_2 relative to the orderings a, b and x, y, z is

$$A_2 = \begin{array}{c|ccc} & x & Y & z \\ \hline a & 1 & 1 & 0 \\ b & 0 & 1 & 1 \end{array}$$

– The product of these matrices is

$$A_1A_2 = \begin{array}{c|ccc} & x & y & z \\ \hline 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{array}$$

Summary

- Equivalence Relations
- Matrices of Relations