

### Discrete Mathematics CS204: Spring, 2008

Jong C. Park Computer Science Division, KAIST **Today's Topics** Equivalence Relations Matrices of Relations

# Relations





#### Theorem

– Let S be a partition of a set X. Define x R y to mean that for some set S in S, both x and y belong to S. Then R is reflexive, symmetric, and transitive.

#### – Proof.

- Let x ∈ X. By the definition of partition, x belongs to some member S of S. Thus x R x and R is reflexive.
- Suppose that x R y. Then both x and y belong to some set S ∈ S. Since both y and x belong to S, y R x and R is symmetric.
- Finally, suppose that x R y and y R z. Then both x and y belong to some set  $S \in S$  and both y and z belong to some set  $T \in S$ . Since y belongs to exactly one member of S, we must have S = T. Therefore, both x and z belong to S and x R z. We have shown that R is transitive.



#### Example

- Consider the partition  $S = \{\{1,3,5\}, \{2,6\}, \{4\}\}\$ of  $X = \{1,2,3,4,5,6\}.$
- What is the relation R on X as given by the preceding theorem?



#### Definition

- A relation that is reflexive, symmetric, and transitive on a set X is called an equivalence relation on X.
- Are any of the following relations equivalence relations?
  - the relation  $R = \{(1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5)\}$
  - the relation R on  $X = \{1,2,3,4\}$  defined by  $(x,y) \in R$  if  $x \le y, x, y \in X$
  - the relation  $R = \{(a,a), (b,c), (c,b), (d,d)\}$  on  $X = \{a,b,c,d\}$



#### Theorem

- Let *R* be an equivalence relation on a set *X*. For each  $a \in X$ , let  $[a] = \{x \in X \mid x R a\}$ . (In words, [a]is the set of all elements in *X* that are related to *a*.) Then  $S = \{[a] \mid a \in X\}$  is a partition of *X*.
- Proof.
  - We must show that every element in X belongs to exactly one member of S. Let a ∈ X. Since a R a, a ∈ [a]. Thus every element in X belongs to at least one member of S.
  - It remains to show that every element in X belongs to exactly one member of S; that is,

if  $x \in X$  and  $x \in [a] \cap [b]$ , then [a] = [b].



#### – Proof (continued).

- We first show that for all  $c, d \in X$ , if c R d, then [c] = [d]. Suppose that c R d. Let  $x \in [c]$ . Then x R c. Since c R d and R is transitive, x R d. Therefore,  $x \in [d]$  and  $[c] \subseteq [d]$ . The argument that  $[d] \subseteq [c]$  is the same as that just given, but with the roles of c and d interchanged. Thus [c] = [d].
- We now prove the following claim.

if  $x \in X$  and  $x \in [a] \cap [b]$ , then [a] = [b]. Assume that  $x \in X$  and  $x \in [a] \cap [b]$ . Then x R aand x R b. Our preceding result shows that [x] = [a]and [x] = [b]. Thus [a] = [b].



#### Definition

 Let R be an equivalence relation on a set X. The sets
 [a] as defined in the preceding theorem are called the equivalence classes of X given by the relation R.

#### Examples

- Given the equivalence relation  $R = \{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5), (2,2), (2,6), (6,2), (6,6), (4,4)\}$  on  $X = \{1,2,3,4,5,6\}$ , determine the equivalence classes [1], [2], [3], and [4].
- What are the equivalence classes for the equivalence relation  $R = \{(1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5)\}$ ?



#### Theorem

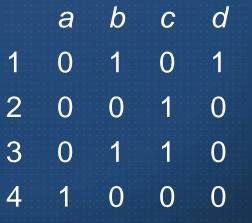
Let *R* be an equivalence relation on a finite set *X*. If each equivalence class has *r* elements, there are |*X*|/*r* equivalence classes.
Proof.

- Let  $X_1, X_2, ..., X_k$  denote the distinct equivalence classes.
- Since these sets partition X,  $|X| = |X_1| + |X_2| + \dots + |X_k| = r + r + \dots + r = kr$  and the conclusion follows.



#### Example

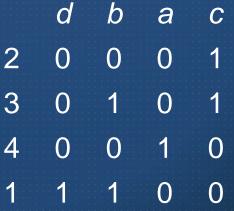
- The matrix of the relation  $R = \{(1,b), (1,d), (2,c), (3,c), (3,b), (4,a)\}$  from  $X = \{1,2,3,4\}$  to  $Y = \{a,b,c,d\}$  relative to the orderings 1, 2, 3, 4 and a, b, c, d is





#### Example

- The matrix of the relation  $R = \{(1,b), (1,d), (2,c), (3,c), (3,b), (4,a)\}$  from  $X = \{1,2,3,4\}$  to  $Y = \{a,b,c,d\}$  relative to the orderings 2, 3, 4, 1 and d, b, a, c is





#### Example

- The matrix of the relation R from {2,3,4} to {5,6,7,8}, relative to the orderings 2, 3, 4 and 5, 6, 7, 8, defined by x R y if x divides y is



#### Example

The matrix of the relation R = {(a,a), (b,b),
 (c,c), (d,d), (b,c), (c,b)} on {a,b,c,d}, relative to the ordering a, b, c, d, is

abcda100b0110c0110d0001



- How to determine whether a relation R on a set X is
  - reflexive?
    - if and only if A has 1's on the main diagonal
    - if and only if  $(x,x) \in R$  for all  $x \in X$
  - symmetric?
    - if and only if A is symmetric about the main diagonal
    - if and only if for all *i* and *j*, the *ij*th entry of A is equal to the *ji*th entry of A
  - antisymmetric?
    - whenever the *ij*th entry is 1,  $i \neq j$ , the *ji*th entry is not 1



#### Example

Let R<sub>1</sub> be the relation from X = {1,2,3} to Y = {a,b} defined by R<sub>1</sub> = {(1,a), (2,b), (3,a), (3,b)}, and let R<sub>2</sub> be the relation from Y to Z = {x,y,z} defined by R<sub>2</sub> = {(a,x), (a,y), (b,y), (b,z)}.
The matrix of R<sub>1</sub> relative to the orderings 1, 2,

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b

0

1

a

1

1

 $A_1 = 2 0 1$ 

3

3 and a, b is



and the matrix of R<sub>2</sub> relative to the orderings
 *a*, *b* and *x*, *y*, *z* is

x Y z

 $A_{2} = \begin{cases} a & 1 & 1 & 0 \\ b & 0 & 1 & 1 \end{cases}$ - The product of these matrices is  $x \quad y \quad z$   $1 \quad 1 \quad 1 \quad 0$   $A_{1}A_{2} = 2 \quad 0 \quad 1 \quad 1$   $3 \quad 1 \quad 2 \quad 1$ 





- Equivalence Relations
- Matrices of Relations

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