Today’s Topics
Introduction
Solving Recurrence Relations
• Definition
  – A **recurrence relation** for the sequence \( a_0, a_1, \ldots \) is an equation that relates \( a_n \) to certain of its predecessors \( a_0, a_1, \ldots, a_{n-1} \).
  – Initial conditions for the sequence \( a_0, a_1, \ldots \) are explicitly given values for a finite number of the terms of the sequence.

• Examples
  – The Fibonacci sequence
    • \( f_n = f_{n-1} + f_{n-2}, \ n \geq 3 \)
    • \( f_1 = 1, \ f_2 = 1 \)
Examples

– Let $S_n$ denote the number of subsets of an $n$-element set. Find its recurrence relation.
  
  \begin{itemize}
    \item $S_n = 2S_{n-1}$
    \item $S_0 = 1$
  \end{itemize}

– Let $S_n$ denote the number of $n$-bit strings that do not contain the pattern 111. Develop a recurrence relation for $S_1$, $S_2$, ... and initial conditions that define the sequence $S$.
  
  \begin{itemize}
    \item Count the number of $n$-bit string that do not contain the pattern 111 (a) that begin with 0; (b) that begin with 10; and (c) that begin with 11.
    \item $S_n = S_{n-1} + S_{n-2} + S_{n-3}$, $n \geq 4$
    \item $S_1 = 2$, $S_2 = 4$, $S_3 = 7$
  \end{itemize}
Example

- Tower of Hanoi
  - The Tower of Hanoi is a puzzle consisting of three pegs mounted on a board and $n$ disks of various sizes with holes in their centers.
    - It is assumed that if a disk is on a peg, only a disk of smaller diameter can be placed on top of the first disk.
    - Given all the disks stacked on one peg, the problem is to transfer the disks to another peg by moving one disk at a time.

- Let $c_n$ denote the number of moves our solution takes to solve the $n$-disk puzzle. Find its recurrence relation.
  - $c_n = 2c_{n-1} + 1$, $n > 1$
  - $c_1 = 1$
Example

- The Cobweb in Economics
  
  Assume an economics model in which the supply and demand are given by linear equations.
  
  Specifically, the demand is given by the equation $p = a - bq$, where $p$ is the price, $q$ is the quantity, and $a$ and $b$ are positive parameters.
  
  The supply is given by the equation $p = kq$, where $p$ is the price, $q$ is the quantity, and $k$ is a positive parameter.
Assume further that there is a time lag as the supply reacts to changes. We denote the discrete time intervals as $n = 0, 1, \ldots$.

Assume that the demand is given by the equation $p_n = a - bq_n$; that is, at time $n$, the quantity $q_n$ of the product will be sold at price $p_n$.

Assume that the supply is given by the equation $p_n = kq_{n+1}$; that is, one unit of time is required for the manufacturer to adjust the quantity $q_{n+1}$, at time $n + 1$, to the price $p_n$, at the prior time $n$.

Solve the equation to obtain a relevant recurrence relation.

$$- p_{n+1} = a - (b/k)p_n$$
Example

- Ackermann’s Function
  - Ackermann’s function can be defined by the recurrence relations
    - $A(m,0) = A(m-1,1), m = 1, 2, \ldots$
    - $A(m,n) = A(m-1,A(m,n-1)), m = 1, 2, \ldots, n = 1, 2, \ldots$
    - $A(0,n) = n + 1, n = 0, 1, \ldots$

- Example
  - $A(1,1) = A(0, A(1,0))$
    - $= A(0, A(0,1))$
    - $= A(0,2)$
    - $= 3$
Examples

- Solve the recurrence relation:
  - $a_n = a_{n-1} + 3$
  - $a_1 = 2$
  - $a_n = a_1 + (n-1) \cdot 3 = 2 + 3(n-1)$

- Solve the recurrence relation:
  - $S_n = 2S_{n-1}$
  - $S_0 = 1$
  - $S_n = 2S_{n-1} = 2(2S_{n-2}) = \ldots = 2^nS_0 = 2^n$
**Solving Recurrence Relations**

- **Definition**
  - A linear homogeneous recurrence relation of order $k$ with constant coefficients is a recurrence relation of the form
    \[ a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}, \ c_k \neq 0. \]

- **Example**
  - \( S_n = 2S_{n-1} \)
  - \( f_n = f_{n-1} + f_{n-2} \)
The following shows examples that are not “linear homogeneous recurrence relations with constant coefficients”.

- $a_n = 3a_{n-1}a_{n-2}$
- $a_n - a_{n-1} = 2n$
- $a_n = 3na_{n-1}$
Solving Recurrence Relations

• Note
  – The general method of solving linear homogeneous recurrence relations with constant coefficients is to find an explicit formula for the sequence defined by the recurrence relation.

• Example
  – Solve the linear homogeneous recurrence relations with constant coefficients
    • $a_n = 5a_{n-1} - 6a_{n-2}$
    • $a_0 = 7, a_1 = 16$
Solution

- Often in mathematics, when trying to solve a more difficult instance of some problem, we begin with an expression that solved a simpler version.

- For the first-order recurrence relation, we found that the solution was of the form \( S_n = t^n \); thus for our first attempt at finding a solution of the second-order recurrence relation, we will search for a solution of the form \( V_n = t^n \).

- If \( V_n = t^n \) is to solve the recurrence relation, we must have \( V_n = 5V_{n-1} - 6V_{n-2} \) or \( t^n = 5t^{n-1} - 6t^{n-2} \) or \( t^n - 5t^{n-1} + 6t^{n-2} = 0 \). Dividing by \( t^{n-2} \), we obtain the equivalent equation \( t^2 - 5t^1 + 6 = 0 \). Solving this, we find the solutions \( t = 2 \), \( t = 3 \).
• We thus have two solutions, $S_n = 2^n$ and $T_n = 3^n$.

• We can verify that if $S$ and $T$ are solutions of the preceding recurrence relation, then $bS + dT$, where $b$ and $d$ are any numbers whatever, is also a solution of that relation. In our case, if we define the sequence $U$ by the equation
  
  $U_n = bS_n + dT_n = b2^n + d3^n$,

  $U$ is a solution of the given relation.

• To satisfy the initial conditions, we must have
  
  $7 = U_0 = b2^0 + d3^0 = b + d$,
  
  $16 = U_1 = b2^1 + d3^1 = 2b + 3d$.

  Solving these equations for $b$ and $d$, we obtain $b = 5$, $d = 2$.

• Therefore, the sequence $U$ defined by $U_n = 5\cdot2^n + 2\cdot3^n$ satisfies the recurrence relation and the initial conditions.

• We conclude that $a_n = U_n = 5\cdot2^n + 2\cdot3^n$, for $n = 0, 1, \ldots$. 
Theorem

- Let \( a_n = c_1 a_{n-1} + c_2 a_{n-2} \) be a second-order, linear homogeneous recurrence relation with constant coefficients.
  
  - If \( S \) and \( T \) are solutions of the recurrence relation, then \( U = bS + dT \) is also a solution of the relation.
  
  - If \( r \) is a root of \( \ell^2 - c_1 \ell - c_2 = 0 \), then the sequence \( r^n, n = 0, 1, \ldots \), is a solution of the recurrence relation.
  
  - If \( a \) is the sequence defined by the recurrence relation, \( a_0 = C_0, a_1 = C_1 \), and \( r_1 \) and \( r_2 \) are roots of the preceding equation with \( r_1 \neq r_2 \), then there exist constants \( b \) and \( d \) such that \( a_n = br_1^n + dr_2^n, n = 0, 1, \ldots \)
Proof.

Since \(S\) and \(T\) are solutions of the relation,
\[
S_n = c_1 S_{n-1} + c_2 S_{n-2}, \quad T_n = c_1 T_{n-1} + c_2 T_{n-2}.
\]

Multiply the first equation by \(b\) and the second by \(d\) and add, to obtain
\[
U_n = bS_n + dT_n
= c_1(bS_{n-1} + dT_{n-1}) + c_2(bS_{n-1} + dT_{n-2})
= c_1 U_{n-1} + c_2 U_{n-2}.
\]

Therefore, \(U\) is a solution of the equation
\[
t^2 - c_1 t - c_2 = 0.
\]
Example

- More Population Growth
  - Assume that the deer population of Rustic County is 200 at time \( n = 0 \) and 220 at time \( n = 1 \) and that the increase from time \( n-1 \) to time \( n \) is twice the increase from time \( n-2 \) to time \( n-1 \).
  - Write a recurrence relation and an initial condition that define the deer population at time \( n \) and then solve the recurrence relation.
Solving Recurrence Relations

- Solution
  
  - Let $d_n$ denote the deer population at time $n$.
    - $d_0 = 200$, $d_1 = 220$.
    - $d_n - d_{n-1} = 2(d_{n-1} - d_{n-2})$.
    - $d_n = 3d_{n-1} - 2d_{n-2}$.
  
  - Solving $t^2 - 3t + 2 = 0$, we have roots 1 and 2. Then the sequence $d$ is of the form $d_n = b \cdot 1^n + c \cdot 2^n = b + c2^n$.
  
  - To meet the initial conditions, we must have $200 = d_0 = b + c$, $220 = d_1 = b + 2c$. Solving for $b$ and $c$, we find $b = 180$, and $c = 20$.
  
  - Thus, $d_n$ is given by $d_n = 180 + 20 \cdot 2^n$. 
Example

– Find an explicit formula for the Fibonacci sequence.
  • $f_n - f_{n-1} - f_{n-2} = 0, \ n \geq 3$.
  • $f_1 = 1, \ f_2 = 1$.

– Solution
  • We begin by using the quadratic formula to solve $t^2 - t - 1 = 0$. The solutions are $t = (1 \pm \sqrt{5})/2$. Thus the solution is of the form $f_n = b((1+\sqrt{5})/2)^n + c((1-\sqrt{5})/2)^n$.
  • To satisfy the initial conditions, we must have $b((1+\sqrt{5})/2) + c((1-\sqrt{5})/2) = 1$, $b((1+\sqrt{5})/2)^2 + c((1-\sqrt{5})/2)^2 = 1$. Solving these equations for $b$ and $d$, we obtain $b = 1/\sqrt{5}, \ d = -1/\sqrt{5}$.
  • Therefore, $f_n = 1/\sqrt{5} \cdot ((1+\sqrt{5})/2)^n - 1/\sqrt{5}((1-\sqrt{5})/2)^n$. 
Solving Recurrence Relations

• Theorem
  – Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ be a second-order, linear homogeneous recurrence relation with constant coefficients.
    • Let $a$ be the sequence satisfying the relation and $a_0 = C_0, a_1 = C_1$.
    • If both roots of $t^2 - c_1 t - c_2 = 0$ are equal to $r$, then there exist constants $b$ and $d$ such that $a_n = br^n + dnr^n, n = 0, 1, \ldots.$
– Proof.

• The proof of the previous theorem shows that the sequence \( r^n \), \( n = 0, 1, \ldots \), is a solution of the relation. We show that the sequence \( nr^n \), \( n = 0, 1, \ldots \), is also a solution of the relation.
  – Since \( r \) is the only solution of the equation, we must have \( t^2 - c_1 t - c_2 = (t - r)^2 \). It follows that \( c_1 = 2r \), \( c_2 = -r^2 \).
  – Now \( a_n = c_1[(n-1)r^{n-1}] + c_2[(n-2)r^{n-2}] \)
    \[ = 2r(n-1)r^{n-1} - r^2(n-2)r^{n-2} \]
    \[ = r^n[2(n-1) - (n-2)] = nr^n \]
  – Therefore, the sequence \( nr^n \), \( n = 0, 1, \ldots \), is a solution of the recurrence relation.

• The sequence \( U \) defined by \( U_n = br^n + dnr^n \) is a solution of the relation. With a similar proof in the previous theorem, there exist constants \( b \) and \( d \) such that \( U_0 = C_0 \) and \( U_1 = C_1 \).

• It follows that \( U_n = a_n \), \( n = 0, 1, \ldots \).
Example

- Solve the recurrence relation \( d_n = 4(d_{n-1} - d_{n-2}) \)
  subject to the initial conditions \( d_0 = 1 = d_1 \).

  • According to the theorem, \( S_n = r^n \) is a solution, where \( r \) is a solution of \( t^2 - 4t + 4 = 0 \). Thus we obtain the solution \( S_n = 2^n \).

  • Since 2 is the only solution of the equation, \( T_n = n2^n \) is also a solution of the recurrence relation.

  • Thus the general solution is of the form \( U = aS + bT \).

  • We must have \( U_0 = 1 = U_1 \). The last equations become \( aS_0 + bT_0 = a + 0b = 1 \), \( aS_1 + bT_1 = 2a + 2b = 1 \).

  • Solving for \( a \) and \( b \), we obtain \( a = 1 \), \( b = -1/2 \).

  • Therefore the solution is \( d_n = 2n - n2^{n-1} \).
• Note
– For the general linear homogeneous recurrence relation of order $k$ with constant coefficients $c_1, c_2, \ldots, c_k$, if $r$ is a root of
  \[ t^k - c_1 t^{k-1} - c_2 t^{k-2} - \ldots - c_k = 0 \]
of multiplicity $m$, it can be shown that
  \[ r^n, nr^n, \ldots, n^{m-1}r^n \]
are solutions of the equation.
Summary

• Recurrence Relations
• Solving Recurrence Relations