Discrete Mathematics

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Today's Topics

Introduction

Solving Recurrence Relations

Recurrence Relations

Introduction

- Definition
  - A recurrence relation for the sequence a_0, a_1, ... is an equation that relates a_n to certain of its predecessors a_0, a_1, ..., a_{n-1}.
  - Initial conditions for the sequence a_0, a_1, ... are explicitly given values for a finite number of the terms of the sequence.
- Examples
  - The Fibonacci sequence
    - $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$
    - $f_1 = 1$, $f_2 = 1$

Introduction

- Examples
  - Let S_n denote the number of subsets of an n-element set. Find its recurrence relation.
    - $S_n = 2S_{n-1}$
    - $S_0 = 1$
  - Let S_n denote the number of n-bit strings that do not contain the pattern 111. Develop a recurrence relation for $S_1, S_2, \ldots$ and initial conditions that define the sequence S.
    - Count the number of n-bit string that do not contain the pattern 111 (a) that begin with 0; (b) that begin with 10; and (c) that begin with 11.
    - $S_n = S_{n-1} + S_{n-2} + S_{n-3}, n \geq 4$
    - $S_1 = 2, S_2 = 4, S_3 = 7$

Introduction

- Example
  - Tower of Hanoi
    - The Tower of Hanoi is a puzzle consisting of three pegs mounted on a board and *n* disks of various sizes with holes in their centers.
      - It is assumed that if a disk is on a peg, only a disk of smaller diameter can be placed on top of the first disk.
      - Given all the disks stacked on one peg, the problem is to transfer the disks to another peg by moving one disk at a time.
    - Let c_n denote the number of moves our solution takes to solve the *n*-disk puzzle. Find its recurrence relation.
      - $c_n = 2c_{n-1} + 1, n > 1$
      - $c_1 = 1$

Introduction

- Example
  - The Cobweb in Economics
    - Assume an economics model in which the supply and demand are given by linear equations.
    - Specifically, the demand is given by the equation $p = a - bq$, where p is the price, q is the quantity, and a and b are positive parameters.
    - The supply is given by the equation $p = kq$, where p is the price, q is the quantity, and k is a positive parameter.

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- Assume further that there is a time lag as the supply reacts to changes. We denote the discrete time intervals as $n = 0, 1, \ldots$.
- Assume that the demand is given by the equation $p_n = a - bq_n$; that is, at time n, the quantity q_n of the product will be sold at price p_n.
- Assume that the supply is given by the equation $p_n = kq_{n+1}$; that is, one unit of time is required for the manufacturer to adjust the quantity q_{n+1}, at time $n + 1$, to the price p_n, at the prior time n.
- Solve the equation to obtain a relevant recurrence relation.
  - $p_{n+1} = a - (b/k)p_n$

Introduction

- Example
  - Ackermann's Function
    - Ackermann's function can be defined by the recurrence relations
      - $A(m,0) = A(m-1,1)$, $m = 1, 2, \ldots$
      - $A(m,n) = A(m-1,A(m,n-1))$, $m = 1, 2, \ldots$, $n = 1, 2, \ldots$
      - $A(0,n) = n + 1$, $n = 0, 1, \ldots$
    - Example
      - $$\begin{aligned} A(1,1) &= A(0, A(1,0)) \\ &= A(0, A(0,1)) \\ &= A(0,2) \\ &= 3 \end{aligned}$$

Solving Recurrence Relations

- Examples
  - Solve the recurrence relation:
    - $a_n = a_{n-1} + 3$
    - $a_1 = 2$

    - $a_n = a_1 + (n-1)\bullet 3 = 2 + 3(n-1)$
  - Solve the recurrence relation:
    - $S_n = 2S_{n-1}$
    - $S_0 = 1$

    - $S_n = 2S_{n-1} = 2(2S_{n-2}) = \ldots = 2^n S_0 = 2^n$

Solving Recurrence Relations

- Definition
  - A linear homogeneous recurrence relation of order k with constant coefficients is a recurrence relation of the form
    $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k},\ c_k \neq 0.$$
- Example
  - $S_n = 2S_{n-1}$
  - $f_n = f_{n-1} + f_{n-2}$

Solving Recurrence Relations

- The following shows examples that are not "linear homogeneous recurrence relations with constant coefficients".
  - $a_n = 3a_{n-1}a_{n-2}$
  - $a_n - a_{n-1} = 2n$
  - $a_n = 3na_{n-1}$

Solving Recurrence Relations

- Note
  - The general method of solving linear homogeneous recurrence relations with constant coefficients is to find an explicit formula for the sequence defined by the recurrence relation.
- Example
  - Solve the linear homogeneous recurrence relations with constant coefficients
    - $a_n = 5a_{n-1} - 6a_{n-2}$
    - $a_0 = 7,\ a_1 = 16$

Solving Recurrence Relations

– Solution

- Often in mathematics, when trying to solve a more difficult instance of some problem, we begin with an expression that solved a simpler version.
- For the first-order recurrence relation, we found that the solution was of the form $S_n = t^n$; thus for our first attempt at finding a solution of the second-order recurrence relation, we will search for a solution of the form $V_n = t^n$.
- If $V_n = t^n$ is to solve the recurrence relation, we must have $V_n = 5V_{n-1} - 6V_{n-2}$ or $t^n = 5t^{n-1} - 6t^{n-2}$ or $t^n - 5t^{n-1} + 6t^{n-2} = 0$.
  Dividing by t^{n-2}, we obtain the equivalent equation $t^2 - 5t^1 + 6 = 0$. Solving this, we find the solutions $t = 2$, $t = 3$.

Solving Recurrence Relations

- We thus have two solutions, $S_n = 2^n$ and $T_n = 3^n$.
- We can verify that if *S* and *T* are solutions of the preceding recurrence relation, then *bS* + *dT*, where *b* and *d* are any numbers whatever, is also a solution of that relation. In our case, if we define the sequence *U* by the equation
  $$U_n = bS_n + dT_n = b2^n + d3^n,$$
  U is a solution of the given relation.
- To satisfy the initial conditions, we must have
  $$7 = U_0 = b2^0 + d3^0 = b + d,$$
  $$16 = U_1 = b2^1 + d3^1 = 2b + 3d.$$
  Solving these equations for *b* and *d*, we obtain *b* = 5, *d* = 2.
- Therefore, the sequence *U* defined by $U_n = 5 \cdot 2^n + 2 \cdot 3^n$ satisfies the recurrence relation and the initial conditions.
- We conclude that $a_n = U_n = 5 \cdot 2^n + 2 \cdot 3^n$, for $n = 0, 1, \ldots$.

Solving Recurrence Relations

- Theorem
  - Let $a_n = c_1a_{n-1} + c_2a_{n-2}$ be a second-order, linear homogeneous recurrence relation with constant coefficients.
    - If S and T are solutions of the recurrence relation, then $U = bS + dT$ is also a solution of the relation.
    - If r is a root of $t^2 - c_1t - c_2 = 0$, then the sequence r^n, $n = 0, 1, \ldots$, is a solution of the recurrence relation.
    - If a is the sequence defined by the recurrence relation, $a_0 = C_0$, $a_1 = C_1$, and r_1 and r_2 are roots of the preceding equation with $r_1 \neq r_2$, then there exist constants b and d such that $a_n = br_1^n + dr_2^n$, $n = 0, 1, \ldots$.

Solving Recurrence Relations

- Proof.
  - Since S and T are solutions of the relation,

    $S_n = c_1 S_{n-1} + c_2 S_{n-2}, \quad T_n = c_1 T_{n-1} + c_2 T_{n-2}.$
  - Multiply the first equation by *b* and the second by *d* and add, to obtain

    $$\begin{aligned} U_n &= bS_n + dT_n \\ &= c_1(bS_{n-1} + dT_{n-1}) + c_2(bS_{n-1} + dT_{n-2}) \\ &= c_1 U_{n-1} + c_2 U_{n-2}. \end{aligned}$$
  - Therefore, U is a solution of the equation

    $t^2 - c_1 t - c_2 = 0.$

Solving Recurrence Relations

- Example
  - More Population Growth
    - Assume that the deer population of Rustic County is 200 at time $n = 0$ and 220 at time $n = 1$ and that the increase from time $n–1$ to time n is twice the increase from time $n–2$ to time $n–1$.
    - Write a recurrence relation and an initial condition that define the deer population at time n and then solve the recurrence relation.

Solving Recurrence Relations

– Solution
  - Let d_n denote the deer population at time n.
    – $d_0 = 200$, $d_1 = 220$.
    – $d_n - d_{n-1} = 2(d_{n-1} - d_{n-2})$.
    – $d_n = 3d_{n-1} - 2d_{n-2}$.
  - Solving $t^2 - 3t + 2 = 0$, we have roots 1 and 2. Then the sequence d is of the form $d_n = b \cdot 1^n + c \cdot 2^n = b + c2^n$.
  - To meet the initial conditions, we must have $200 = d_0 = b + c$, $220 = d_1 = b + 2c$. Solving for b and c, we find $b = 180$, and $c = 20$.
  - Thus, d_n is given by $d_n = 180 + 20 \cdot 2^n$.

Solving Recurrence Relations

- Example
  - Find an explicit formula for the Fibonacci sequence.
    - $f_n - f_{n-1} - f_{n-2} = 0, \;\; n \geq 3$.
    - $f_1 = 1, f_2 = 1$.
  - Solution
    - We begin by using the quadratic formula to solve $t^2 - t - 1 = 0$. The solutions are $t = (1 \pm \sqrt{5})/2$. Thus the solution is of the form $f_n = b((1+\sqrt{5})/2)^n + c((1-\sqrt{5})/2)^n$.
    - To satisfy the initial conditions, we must have $b((1+\sqrt{5})/2) + c((1-\sqrt{5})/2) = 1$, $b((1+\sqrt{5})/2)^2 + c((1-\sqrt{5})/2)^2 = 1$. Solving these equations for b and d, we obtain $b = 1/\sqrt{5}$, d = $-1/\sqrt{5}$.
    - Therefore, $f_n = 1/\sqrt{5}\cdot((1+\sqrt{5})/2)^n - 1/\sqrt{5}((1-\sqrt{5})/2)^n$.

Solving Recurrence Relations

- Theorem
  - Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ be a second-order, linear homogeneous recurrence relation with constant coefficients.
    - Let a be the sequence satisfying the relation and $a_0 = C_0$, $a_1 = C_1$.
    - If both roots of $t^2 - c_1 t - c_2 = 0$ are equal to r, then there exist constants b and d such that $a_n = br^n + dnr^n$, $n = 0, 1, \ldots$.

Solving Recurrence Relations

– Proof.

- The proof of the previous theorem shows that the sequence r^n , $n = 0, 1, \ldots$, is a solution of the relation. We show that the sequence nr^n, $n = 0, 1, \ldots$, is also a solution of the relation.
  - Since r is the only solution of the equation, we must have $t^2 - c_1 t - c_2 = (t - r)^2$. It follows that $c_1 = 2r$, $c_2 = -r^2$.
  - Now $a_n = c_1[(n-1)r^{n-1}] + c_2[(n-2)r^{n-2}]$

    $= 2r(n-1)r^{n-1} - r^2(n-2)r^{n-2}$

    $= r^n[2(n-1) - (n-2)] = nr^n$
  - Therefore, the sequence nr^n, $n = 0, 1, \ldots$, is a solution of the recurrence relation.
- The sequence U defined by $U_n = br^n + dnr^n$ is a solution of the relation. With a similar proof in the previous theorem, there exist constants b and d such that $U_0 = C_0$ and $U_1 = C_1$.
- It follows that $U_n = a_n$, $n = 0, 1, \ldots$.

Solving Recurrence Relations

- Example
  – Solve the recurrence relation $d_n = 4(d_{n-1} – d_{n-2})$ subject to the initial conditions $d_0 = 1 = d_1$.
    - According to the theorem, $S_n = r^n$ is a solution, where r is a solution of $t^2 – 4t + 4 = 0$. Thus we obtain the solution $S_n = 2^n$.
    - Since 2 is the only solution of the equation, $T_n = n2^n$ is also a solution of the recurrence relation.
    - Thus the general solution is of the form $U = aS + bT$.
    - We must have $U_0 = 1 = U_1$. The last equations become $aS_0 + bT_0 = a + 0b = 1$, $aS_1 + bT_1 = 2a + 2b = 1$.
    - Solving for a and b, we obtain $a = 1$, $b = -1/2$.
    - Therefore the solution is $d_n = 2n – n2^{n-1}$.

Solving Recurrence Relations

- Note
  - For the general linear homogeneous recurrence relation of order k with constant coefficients $c_1, c_2, \ldots, c_k$,
    if r is a root of
    $$t^k - c_1 t^{k-1} - c_2 t^{k-2} - \ldots - c_k = 0$$
    of multiplicity m, it can be shown that
    $$r^n, nr^n, \ldots, n^{m-1} r^n$$
    are solutions of the equation.

Summary

- Recurrence Relations
- Solving Recurrence Relations