Discrete Mathematics

CS204: Spring, 2008

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Today's Topics

- Introduction
- Paths and Cycles
- Hamiltonian Cycles and the Traveling Salesperson Problem
- A Shortest-Path Algorithm
- Representations of Graphs
- Isomorphisms of Graphs
- Planar Graphs

Graph Theory

Introduction

- Definition
  - A graph (or undirected graph) G consists of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge $e \in E$ is associated with an unordered pair of vertices.
  - If there is a unique edge e associated with the vertices v and w, we write $e = (v,w)$ or $e = (w,v)$.
    - In this context, (v,w) denotes an edge between v and w in an undirected graph and not an ordered pair.

Introduction

- A directed graph (or digraph) G consists of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge $e \in E$ is associated with the ordered pair (v,w) of vertices.
  - If there is a unique edge e associated with the ordered pair (v,w) of vertices, we write $e = (v,w)$, which denotes an edge from v to w.
- An edge e in a graph (undirected or directed) that is associated with the pair of vertices v and w is said to be incident on v and w, and v and w are said to be incident on e and to be adjacent vertices.

Introduction

– If G is a graph (undirected or directed) with vertices V and edges E, we write $G = (V,E)$.
– Unless specified otherwise, the sets E and V are assumed to be finite and V is assumed to be nonempty.

Introduction

- Example
  - Find the (undirected) graph G from the part of the Wyoming highway system.
    - V = {Gre, She, Wor, Buf, Gil, Sho, Cas, Dou, Lan, Mud}
    - $E = \{e_1, e_2, ..., e_{13}\}$
- Note
  - In a directed graph, the directed edges are indicated by arrows. (cf. Figure 8.1.4)

Introduction

- Note
  - parallel edges
    - edges that are associated with the same vertex pair
  - loop
    - an edge incident on a single vertex
  - isolated vertex
    - a vertex that is not incident on any edge
  - simple graph
    - a graph with neither loops nor parallel edges
- Example
  - Figure 8.1.2

Introduction

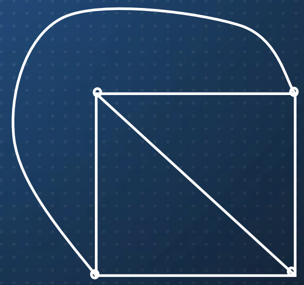
- Example
  - Manufacturing problem
    - A graph with numbers on the edges is called a weighted graph.
    - If edge *e* is labeled *k*, we say that the weight of edge *e* is *k*.
    - In a weighted graph, the length of a path is the sum of the weights of the edges in the path.
    - A path of minimum length that visits every vertex exactly one time represents the optimal path for the drill press to follow.

Introduction

- Examples
  - Bacon Numbers
    - Actor Kevin Bacon
  - Similarity Graphs (of a program)
    - The number of lines
    - The number of return statements
    - The number of function calls
  - The *n*-Cube (Hypercube)

Introduction

- Definition
  - The complete graph on n vertices, denoted K_n, is the simple graph with n vertices in which there is an edge between every pair of distinct vertices.
- Example
  - the complete graph on four vertices, K_4



Introduction

- Definition
  - A graph $G = (V,E)$ is bipartite if there exist subsets V_1 and V_2 (either possibly empty) of V such that $V_1 \cap V_2 = \varnothing$, $V_1 \cup V_2 = V$, and each edge in E is incident on one vertex in V_1 and one vertex in V_2.
- Examples
  - Are the following graphs bipartite?
    - the graph in Figure 8.1.13
    - the graph in Figure 8.1.14
    - the complete graph K_1 on one vertex

Introduction

- Definition
  - The complete bipartite graph on m and n vertices, denoted $K_{m,n}$, is the simple graph whose vertex set is partitioned into sets V_1 with m vertices and V_2 with n vertices in which the edge set consists of all edges of the form (v_1,v_2) with $v_1 \in V_1$ and $v_2 \in V_2$.
- Example
  - the complete bipartite graph on two and four vertices, $K_{2,4}$

Paths and Cycles

- Definition
  - Let v_0 and v_n be vertices in a graph.
  - A path from v_0 to v_n of length n is an alternating sequence of n+1 vertices and n edges beginning with vertex v_0 and ending with vertex v_n, $(v_0,e_1,v_1,e_2,v_2,\ldots,v_{n-1},e_n,v_n)$, in which edge e_i is incident on vertices v_{i-1} and v_i for $i = 1, \ldots, n$.
- Example
  - Find a path of length 4 from vertex 1 to vertex 2 in the graph of Figure 8.2.1.

Paths and Cycles

- Note
  - In the absence of parallel edges, in denoting a path we may suppress the edges.
    - Example: (1,2,3,4,2)
- Definition
  - A graph *G* is connected if given any vertices *v* and *w* in *G*, there is a path from *v* to *w*.
- Examples
  - Are the following graphs connected?
    - the graph *G* of Figure 8.2.1
    - the graph *G* of Figure 8.2.2

Paths and Cycles

- Definition
  - Let $G = (V,E)$ be a graph.
  - We call (V',E') a subgraph of G if
    (a) $V' \subseteq V$ and $E' \subseteq E$.
    (b) For every edge $e' \in E'$, if e' is incident on v' and w', then v', $w' \in V'$.
- Examples
  - Is the graph $G' = (V',E')$ of Figure 8.2.3 a subgraph of the graph $G = (V,E)$ of Figure 8.2.4?
  - Find all subgraphs of the graph G of Figure 8.2.5 having at least one vertex.

Paths and Cycles

- Definition
  - Let G be a graph and let v be a vertex in G.
  - The subgraph G' of G consisting of all edges and vertices in G that are contained in some path beginning at v is called the component of G containing v.
- Examples
  - Find the component(s) of the graph G of Figure 8.2.1.
  - Let G be the graph of Figure 8.2.2.
    - Find the component of G containing v_3.
      - the subgraph $G_1 = (V_1,E_1)$, $V_1 = \{v_1,v_2,v_3\}$, $E_1 = \{e_1,e_2,e_3\}$.
    - Find the component of G containing v_4.
    - Find the component of G containing v_5.

Paths and Cycles

- Definition
  - Let *v* and *w* be vertices in a graph *G*.
    - A simple path from *v* to *w* is a path from *v* to *w* with no repeated vertices.
    - A cycle (or circuit) is a path of nonzero length from *v* to *v* with no repeated edges.
    - A simple cycle is a cycle from *v* to *v* in which, except for the beginning and ending vertices that are both equal to *v*, there are no repeated vertices.
- Example
  - the graph of Figure 8.2.1

Paths and Cycles

- Example
  - Königsberg Bridge Problem
- Note
  - an Euler cycle
    - a cycle in a graph G that includes all of the edges and all of the vertices of G
  - the degree of a vertex v, ($\delta(v)$)
    - the number of edges incident on v

Paths and Cycles

- Theorem
  - If a graph G has an Euler cycle, then G is connected and every vertex has even degree.
- Theorem
  - If G is a connected graph and every vertex has even degree, then G has an Euler cycle.
  - Proof.
    - The proof is by induction on the number n of edges in G.

Paths and Cycles

- Example
  - Let G be the graph of Figure 8.2.10.
    - Use Theorem 8.2.18 to verify that G has an Euler cycle.
    - Find an Euler cycle for G.

Paths and Cycles

- Example
  - A domino is a rectangle divided into two squares with each square numbered one of 0, 1, ..., 6. Two squares on a single domino can have the same number.
  - Show that distinct dominoes can be arranged in a circle so that touching dominoes have adjacent squares with identical numbers.
    - We model the situation as a graph G with seven vertices labeled 0, 1, ..., 6. The edges represent the dominoes: There is one edge between each distinct pair of vertices and there is one loop at each vertex. Notice that G is connected.

Paths and Cycles

- The dominoes can be arranged in a circle so that touching dominoes have adjacent squares with identical numbers if and only if G contains an Euler cycle.
- Since the degree of each vertex is 8, each vertex has even degree. By Theorem 8.2.18, G has an Euler cycle.
- Therefore, the dominoes can be arranged in a circle so that touching dominoes have adjacent squares with identical numbers.

Paths and Cycles

- Theorem
  - If G is a graph with m edges and vertices $\{v_1, v_2, ..., v_n\}$, then $\Sigma^n_{i=1}\, \delta(v_i) = 2m$. In particular, the sum of the degrees of all the vertices in a graph is even.
- Corollary
  - In any graph, there are an even number of vertices of odd degree.

Paths and Cycles

- Theorem
  - A graph has a path with no repeated edges from v to w ($v \neq w$) containing all the edges and vertices if and only if it is connected and v and w are the only vertices having odd degree.
- Theorem
  - If a graph G contains a cycle from v to v, G contains a simple cycle from v to v.

Summary

- Paths and Cycles
- Hamiltonian Cycles and the Traveling Salesperson Problem
- A Shortest-Path Algorithm
- Representations of Graphs
- Isomorphisms of Graphs
- Planar Graphs