Discrete Mathematics
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Today’s Topics
Introduction
Paths and Cycles
Hamiltonian Cycles and the Traveling Salesperson Problem
A Shortest-Path Algorithm
Representations of Graphs
Isomorphisms of Graphs
Planar Graphs

Graph Theory
Definition

- A graph (or undirected graph) $G$ consists of a set $V$ of vertices (or nodes) and a set $E$ of edges (or arcs) such that each edge $e \in E$ is associated with an unordered pair of vertices.

- If there is a unique edge $e$ associated with the vertices $v$ and $w$, we write $e = (v, w)$ or $e = (w, v)$.

  - In this context, $(v, w)$ denotes an edge between $v$ and $w$ in an undirected graph and not an ordered pair.
A directed graph (or digraph) \( G \) consists of a set \( V \) of vertices (or nodes) and a set \( E \) of edges (or arcs) such that each edge \( e \in E \) is associated with the ordered pair \((v, w)\) of vertices.

- If there is a unique edge \( e \) associated with the ordered pair \((v, w)\) of vertices, we write \( e = (v, w) \), which denotes an edge from \( v \) to \( w \).

- An edge \( e \) in a graph (undirected or directed) that is associated with the pair of vertices \( v \) and \( w \) is said to be incident on \( v \) and \( w \), and \( v \) and \( w \) are said to be incident on \( e \) and to be adjacent vertices.
If $G$ is a graph (undirected or directed) with vertices $V$ and edges $E$, we write $G = (V,E)$.

Unless specified otherwise, the sets $E$ and $V$ are assumed to be finite and $V$ is assumed to be nonempty.
Example
- Find the (undirected) graph $G$ from the part of the Wyoming highway system.
  - $V = \{\text{Gre, She, Wor, Buf, Gil, Sho, Cas, Dou, Lan, Mud}\}$
  - $E = \{e_1, e_2, \ldots, e_{13}\}$

Note
- In a directed graph, the directed edges are indicated by arrows. (cf. Figure 8.1.4)
Introduction

• Note
  – parallel edges
    • edges that are associated with the same vertex pair
  – loop
    • an edge incident on a single vertex
  – isolated vertex
    • a vertex that is not incident on any edge
  – simple graph
    • a graph with neither loops nor parallel edges

• Example
  – Figure 8.1.2
• Example
  – Manufacturing problem
    • A graph with numbers on the edges is called a weighted graph.
    • If edge $e$ is labeled $k$, we say that the weight of edge $e$ is $k$.
    • In a weighted graph, the length of a path is the sum of the weights of the edges in the path.
    • A path of minimum length that visits every vertex exactly one time represents the optimal path for the drill press to follow.
Introduction

• Examples
  – Bacon Numbers
    • Actor Kevin Bacon
  – Similarity Graphs (of a program)
    • The number of lines
    • The number of return statements
    • The number of function calls
  – The $n$-Cube (Hypercube)
• Definition
  – The complete graph on \( n \) vertices, denoted \( K_n \), is the simple graph with \( n \) vertices in which there is an edge between every pair of distinct vertices.

• Example
  – the complete graph on four vertices, \( K_4 \)
• **Definition**
  
  - A graph $G = (V,E)$ is **bipartite** if there exist subsets $V_1$ and $V_2$ (either possibly empty) of $V$ such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V$, and each edge in $E$ is incident on one vertex in $V_1$ and one vertex in $V_2$.

• **Examples**
  
  - Are the following graphs bipartite?
    
    - the graph in Figure 8.1.13
    - the graph in Figure 8.1.14
    - the complete graph $K_1$ on one vertex
Introduction

• Definition
  – The complete bipartite graph on \( m \) and \( n \) vertices, denoted \( K_{m,n} \), is the simple graph whose vertex set is partitioned into sets \( V_1 \) with \( m \) vertices and \( V_2 \) with \( n \) vertices in which the edge set consists of all edges of the form \((v_1, v_2)\) with \( v_1 \in V_1 \) and \( v_2 \in V_2 \).

• Example
  – the complete bipartite graph on two and four vertices, \( K_{2,4} \)
Paths and Cycles

• Definition
  – Let $v_0$ and $v_n$ be vertices in a graph.
  – A path from $v_0$ to $v_n$ of length $n$ is an alternating sequence of $n+1$ vertices and $n$ edges beginning with vertex $v_0$ and ending with vertex $v_n$, $(v_0, e_1, v_1, e_2, v_2, ..., v_{n-1}, e_n, v_n)$, in which edge $e_i$ is incident on vertices $v_{i-1}$ and $v_i$ for $i = 1, ..., n$.

• Example
  – Find a path of length 4 from vertex 1 to vertex 2 in the graph of Figure 8.2.1.
Note
- In the absence of parallel edges, in denoting a path we may suppress the edges.
  * Example: (1,2,3,4,2)

Definition
- A graph $G$ is connected if given any vertices $v$ and $w$ in $G$, there is a path from $v$ to $w$.

Examples
- Are the following graphs connected?
  * the graph $G$ of Figure 8.2.1
  * the graph $G$ of Figure 8.2.2
• **Definition**
  - Let $G = (V, E)$ be a graph.
  - We call $(V', E')$ a **subgraph** of $G$ if
    (a) $V' \subseteq V$ and $E' \subseteq E$.
    (b) For every edge $e' \in E'$, if $e'$ is incident on $v'$ and $w'$, then $v', w' \in V'$.

• **Examples**
  - Is the graph $G' = (V', E')$ of Figure 8.2.3 a subgraph of the graph $G = (V, E)$ of Figure 8.2.4?
  - Find all subgraphs of the graph $G$ of Figure 8.2.5 having at least one vertex.
**Definition**

- Let $G$ be a graph and let $v$ be a vertex in $G$.
- The subgraph $G'$ of $G$ consisting of all edges and vertices in $G$ that are contained in some path beginning at $v$ is called the component of $G$ containing $v$.

**Examples**

- Find the component(s) of the graph $G$ of Figure 8.2.1.
- Let $G$ be the graph of Figure 8.2.2.
  - Find the component of $G$ containing $v_3$.
    - the subgraph $G_1 = (V_1, E_1)$, $V_1 = \{v_1, v_2, v_3\}$, $E_1 = \{e_1, e_2, e_3\}$.
  - Find the component of $G$ containing $v_4$.
  - Find the component of $G$ containing $v_5$. 
• Definition
  – Let \( v \) and \( w \) be vertices in a graph \( G \).
    • A simple path from \( v \) to \( w \) is a path from \( v \) to \( w \) with no repeated vertices.
    • A cycle (or circuit) is a path of nonzero length from \( v \) to \( v \) with no repeated edges.
    • A simple cycle is a cycle from \( v \) to \( v \) in which, except for the beginning and ending vertices that are both equal to \( v \), there are no repeated vertices.

• Example
  – the graph of Figure 8.2.1
Paths and Cycles

• Example
  – Königsberg Bridge Problem

• Note
  – an Euler cycle
    • a cycle in a graph $G$ that includes all of the edges and all of the vertices of $G$
  – the degree of a vertex $v$, $(\delta(v))$
    • the number of edges incident on $v$
• Theorem
  – If a graph $G$ has an Euler cycle, then $G$ is connected and every vertex has even degree.

• Theorem
  – If $G$ is a connected graph and every vertex has even degree, then $G$ has an Euler cycle.
  – Proof.
    • The proof is by induction on the number $n$ of edges in $G$. 
Example

- Let $G$ be the graph of Figure 8.2.10.
  - Use Theorem 8.2.18 to verify that $G$ has an Euler cycle.
  - Find an Euler cycle for $G$. 

Paths and Cycles
**Example**

- A domino is a rectangle divided into two squares with each square numbered one of 0, 1, ..., 6. Two squares on a single domino can have the same number.

- Show that distinct dominoes can be arranged in a circle so that touching dominoes have adjacent squares with identical numbers.

  - We model the situation as a graph $G$ with seven vertices labeled 0, 1, ..., 6. The edges represent the dominoes: There is one edge between each distinct pair of vertices and there is one loop at each vertex. Notice that $G$ is connected.
The dominoes can be arranged in a circle so that touching dominoes have adjacent squares with identical numbers if and only if $G$ contains an Euler cycle.

Since the degree of each vertex is 8, each vertex has even degree. By Theorem 8.2.18, $G$ has an Euler cycle.

Therefore, the dominoes can be arranged in a circle so that touching dominoes have adjacent squares with identical numbers.
Theorem
- If $G$ is a graph with $m$ edges and vertices $\{v_1, v_2, ..., v_n\}$, then $\sum_{i=1}^{n} \delta(v_i) = 2m$. In particular, the sum of the degrees of all the vertices in a graph is even.

Corollary
- In any graph, there are an even number of vertices of odd degree.
Paths and Cycles

- Theorem
  - A graph has a path with no repeated edges from $v$ to $w$ ($v \neq w$) containing all the edges and vertices if and only if it is connected and $v$ and $w$ are the only vertices having odd degree.

- Theorem
  - If a graph $G$ contains a cycle from $v$ to $v$, $G$ contains a simple cycle from $v$ to $v$. 
Summary

• Paths and Cycles
• Hamiltonian Cycles and the Traveling Salesperson Problem
• A Shortest-Path Algorithm
• Representations of Graphs
• Isomorphisms of Graphs
• Planar Graphs