Discrete Mathematics

CS204: Spring, 2008

Jong C. Park

Computer Science Division, KAIST

Today’s Topics

Introduction

Paths and Cycles

Hamiltonian Cycles and the Traveling Salesperson Problem

A Shortest-Path Algorithm

Representations of Graphs

Isomorphisms of Graphs

GRAPH THEORY

Hamiltonian Cycles and the Traveling Salesperson Problem

- Hamiltonian cycle
  - a cycle in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice.
- Examples
  - Determine if the following graphs have a Hamiltonian cycle.
    - the graph of Figure 8.3.4
    - the graph of Figure 8.3.5
    - the graph of Figure 8.3.6
    - the graph of Figure 8.3.7

Hamiltonian Cycles and the Traveling Salesperson Problem

- Traveling salesperson problem
  - Given a weighted graph G, find a minimum-length Hamiltonian cycle in G.
- Example
  - the graph of Figure 8.3.8

Hamiltonian Cycles and the Traveling Salesperson Problem

- Models for parallel computation
  - ring model
  - n-cube
- Problem
  - When can an n-cube simulate a ring model with 2^n processors?
  - Equivalently, when does an n-cube contain a Hamiltonian cycle?

Hamiltonian Cycles and the Traveling Salesperson Problem

- Note
  - The n-cube has a Hamiltonian cycle if and only if $n \geq 2$ and there is a sequence, $s_1, s_2, \ldots, s_{2^n}$, where each s_i is a string of n-bits, satisfying:
    - Every n-bit string appears somewhere in the sequence.
    - s_i and s_{i+1} differ in exactly one bit, $i = 1, \ldots, 2^n-1$.
    - s_{2^n} and s_1 differ in exactly one bit.
  - The sequence above is called a Gray code.
  - When $n \geq 2$, a Gray code corresponds to the Hamiltonian cycle $s_1, s_2, \ldots, s_{2^n}, s_1$.

Hamiltonian Cycles and the Traveling Salesperson Problem

- Theorem
  - Let G_1 denote the sequence 0, 1. We define G_n in terms of G_{n-1} by the following rules:
    - (a) Let G^R_{n-1} denote the sequence G_{n-1} written in reverse.
    - (b) Let G_{n-1} denote the sequence obtained by prefixing each member of G_{n-1} with 0.
    - (c) Let G''_{n-1} denote the sequence obtained by prefixing each member of G^R_{n-1} with 1.
    - (d) Let G_n be the sequence consisting of G_{n-1} followed by G''_{n-1}.

    Then G_n is a Gray code for every positive integer n.
  - Proof.
    - The proof is done by induction on n.

Hamiltonian Cycles and the Traveling Salesperson Problem

- Corollary
  - The n-cube has a Hamiltonian cycle for every positive integer $n \geq 2$.
- Examples
  - Construct the Gray code G_3 beginning with G_1
  - The Knight's Tour
    - A knight's tour of an n x n board begins at some square, visits each square exactly once making legal moves, and returns to the initial square.
    - The problem is to determine for which n a knight's tour exists.

A Shortest-Path Algorithm

Algorithm 8.4.1: Dijkstra's Shortest-Path Algorithm

Input: A connected, weighted graph in which all weights are positive; vertices a and z

Output: $L(z)$, the length of a shortest path from a to z

```
dijkstra(w, a, z, L) {
  L(a) = 0
  for all vertices x ≠ a
    L(x) = ∞
  T = set of all vertices
  // T is the set of vertices whose shortest
  // distance from a has not been found
  while (z ∈ T) {
    choose v ∈ T with minimum L(v)
    T = T − {v}
    for each x ∈ T adjacent to v
      L(x) = min{L(x), L(v) + w(v, x)}
  }
}
```

A Shortest-Path Algorithm

- Theorem
  - Dijkstra's shortest-path algorithm correctly finds the length of a shortest path from a to z.
- Example
  - Find a shortest path from a to z and its length for the graph of Figure 8.4.7.
- Theorem
  - For input consisting of an n-vertex, simple, connected, weighted graph, Dijkstra's algorithm has worst-case run time $\Theta(n^2)$.

Representations of Graphs

- Adjacency Matrix
  - Select an ordering of the vertices, say *a*, *b*, *c*, *d*, *e*.
  - Label the rows and columns of a matrix with the ordered vertices.
  - The entry in this matrix in row *i*, column *j*, $i \neq j$, is the number of edges incident on *i* and *j*. If $i = j$, the entry is twice the number of loops incident on *i*.
  - The degree of a vertex *v* in a graph *G* is obtained by summing row *v* or column *v* in *G*s adjacency matrix.

Representations of Graphs

- Theorem
  - If A is the adjacency matrix of a simple graph, the ijth entry of A^n is equal to the number of paths of length n from vertex i to vertex j, $n = 1, 2, \ldots$.
  - Proof.
    - Use induction on n.

Representations of Graphs

- Incidence Matrix
  - We label the rows with the vertices and the columns with the edges (in some arbitrary order).
  - The entry for row v and column e is 1 if e is incident on v and 0 otherwise.
- Example
  - Find the incidence matrix for the graph of Figure 8.5.4.
- Note
  - In a graph without loops, each column has two 1's and the sum of a row gives the degree of the vertex identified with that row.

Isomorphisms of Graphs

- Definition
  - Graphs G_1 and G_2 are isomorphic if there is a one-to-one, onto function f from the vertices of G_1 to the vertices of G_2 and a one-to-one, onto function g from the edges of G_1 to the edges of G_2, so that an edge e is incident on v and w in G_1 if and only if the edge $g(e)$ is incident on $f(v)$ and $f(w)$ in G_2.
  - The pair of functions f and g is called an isomorphism of G_1 onto G_2.
- Example
  - Define an isomorphism for the graphs G_1 and G_2 of Figure 8.6.1.

Isomorphisms of Graphs

- Example
  - The Mesh Model for Parallel Computation
    - The two dimensional mesh model for parallel computation when described as a graph consists of a rectangular array of connected vertices.
  - Problem
    - When can an n-cube simulate a two-dimensional mesh?
    - When does an n-cube contain a subgraph isomorphic to a two-dimensional mesh?
  - Answer
    - If M is a mesh p vertices by q vertices, where $p \leq 2^i$ and $q \leq 2^j$, then the (i+j)-cube contains a subgraph isomorphic to M.

Isomorphisms of Graphs

- Theorem
  – Graphs G_1 and G_2 are isomorphic if and only if for some ordering of their vertices, their adjacency matrices are equal.
- Corollary
  – Let G_1 and G_2 be simple graphs. The following are equivalent.
    (a) G_1 and G_2 are isomorphic.
    (b) There is a one-to-one, onto function f from the vertex set of G_1 to the vertex set of G_2 satisfying the following: Vertices v and w are adjacent in G_1 if and only if the vertices $f(v)$ and $f(w)$ are adjacent in G_2.

Isomorphisms of Graphs

- Example
  - Determine if G_1 and G_2 in Figure 8.6.1 are isomorphic by examining their adjacency matrices.
- Note
  - A property P is an invariant if whenever G_1 and G_2 are isomorphic graphs:

    If G_1 has property P, G_2 also has property P.
  - Examples
    - "has e edges"
    - "has n vertices"

Isomorphisms of Graphs

- Example
  - Use the notion of an invariant to determine if the graphs G_1 and G_2 in Figure 8.6.3 are isomorphic.

Isomorphisms of Graphs

- Examples
  - Show that if k is a positive integer, "has a vertex of degree k" is an invariant.
    - Proof sketch.
      - Suppose G_1 and G_2 are isomorphic graphs and f (resp., g) is a one-to-one, onto function from the vertices (resp., edges) of G_1 onto the vertices (resp., edges) of G_2.
      - Suppose further that G_1 has a vertex v of degree k.
  - Use the fact that "has a vertex of degree 3" is an invariant to determine if the graphs G_1 and G_2 in Figure 8.6.4 are isomorphic.

Isomorphisms of Graphs

- Example
  - Show that if k is a positive integer, "has a simple cycle of length k" is an invariant.
    - Proof.
      - exercise
  - Use the fact that "has a simple cycle of length 3" is an invariant to determine if the graphs G_1 and G_2 of Figure 8.6.5 are isomorphic.

Summary

- Paths and Cycles
- Hamiltonian Cycles and the Traveling Salesperson Problem
- A Shortest-Path Algorithm
- Representations of Graphs
- Isomorphisms of Graphs