Today’s Topics

Combinatorial Circuits
Properties of Combinatorial Circuits
Boolean Algebras
Boolean Functions and Synthesis of Circuits
Applications
Combinatorial Circuits

• Definition
  – An **AND gate** receives inputs $x_1$ and $x_2$, where $x_1$ and $x_2$ are bits, and produces output denoted $x_1 \land x_2$, where $x_1 \land x_2 = 1$ if $x_1 = 1$ and $x_2 = 1$, 0 otherwise.
  – An **OR gate** receives inputs $x_1$ and $x_2$, where $x_1$ and $x_2$ are bits, and produces output denoted $x_1 \lor x_2$, where $x_1 \lor x_2 = 1$ if $x_1 = 1$ or $x_2 = 1$, 0 otherwise.
  – A **NOT gate** (or inverter) receive input $x$, where $x$ is a bit, and produces output denoted $\sim x$, where $\sim x = 1$ if $x = 1$, 0 if $x = 1$. 
Combinatorial Circuits

• Note
  – The logic table of a combinatorial circuit lists all possible inputs together with the resulting outputs.

• Examples
  – logic tables for the basic AND, OR, and NOT circuits
  – the combinatorial circuit in Figure 11.1.4
  – the non-combinatorial circuit in Figure 11.1.6
  – the interconnected combinatorial circuit in Figure 11.1.7
Combinatorial Circuits

• Definition
  – Boolean expressions in the symbols \( x_1, \ldots, x_n \) are defined recursively as follows.
    • 0, 1, \( x_1, \ldots, x_n \) are Boolean expressions.
    • If \( X_1 \) and \( X_2 \) are Boolean expressions, then (a) \((X_1)\), (b) \(~X_1\), (c) \(X_1 \lor X_2\), (d) \(X_1 \land X_2\) are Boolean expressions.
  – If \( X \) is a Boolean expression in the symbols \( x_1, \ldots, x_n \), we sometimes write \( X = X(x_1, \ldots, x_n) \).
  – Either symbol \( x \) or \(~x\) is called a literal.
Combinatorial Circuits

• Examples
  – Computing the value of a Boolean expression
  – Finding the combinatorial circuit corresponding to the Boolean expression \((x_1 \land (\sim x_2 \lor x_3)) \lor x_2\)
Properties of Combinatorial Circuits

• Theorem
  – If $\land$, $\lor$, and $\neg$ are as in the previous definitions, then the following properties hold, where $\mathbb{Z}_2 = \{0, 1\}$.
    
    (a) Associative laws
    – $(a \lor b) \lor c = a \lor (b \lor c)$
    – $(a \land b) \land c = a \land (b \land c)$ for all $a, b, c \in \mathbb{Z}_2$.

    (b) Commutative laws
    – $a \lor b = b \lor a$
    – $a \land b = b \land a$, for all $a, b \in \mathbb{Z}_2$.

    (c) Distributive laws
    – $a \land (b \lor c) = (a \land b) \lor (a \land c)$
    – $a \lor (b \land c) = (a \lor b) \land (a \lor c)$ for all $a, b, c \in \mathbb{Z}_2$. 
Properties of Combinatorial Circuits

(d) Identity laws
   – \( a \lor 0 = a \)
   – \( a \land 1 = a \) for all \( a \in \mathbb{Z}_2 \).

(e) Complement laws
   – \( a \lor \neg a = 1 \)
   – \( a \land \neg a = 0 \) for all \( a \in \mathbb{Z}_2 \).
Properties of Combinatorial Circuits

• Definition
  – Let $X_1 = X_1(x_1, \ldots, x_n)$ and $X_2 = X_2(x_1, \ldots, x_n)$ be Boolean expressions.
  – We define $X_1$ to be equal to $X_2$ and write $X_1 = X_2$ if $X_1(a_1,\ldots,a_n) = X_2(a_1,\ldots,a_n)$ for all $a_i \in \mathbb{Z}_2$.

• Example
  – Show that $\neg(x \lor y) = \neg x \land \neg y$. 
Properties of Combinatorial Circuits

• Definition
  – We say that two combinatorial circuits, each having inputs $x_1, \ldots, x_n$ and a single output, are equivalent if, whenever the circuits receive the same inputs, they produce the same outputs.

• Example
  – Equivalence of the combinatorial circuits of Figures 11.2.4 and 11.2.5
Properties of Combinatorial Circuits

• Theorem
  – Let $C_1$ and $C_2$ be combinatorial circuits represented, respectively, by the Boolean expressions $X_1 = X_1(x_1,\ldots,x_n)$ and $X_2 = X_2(x_1,\ldots,x_n)$.
  – Then $C_1$ and $C_2$ are equivalent if and only if $X_1 = X_2$. 
Boolean Algebras

• Definition
  – A **Boolean algebra** $B$ consists of a set $S$ containing distinct elements 0 and 1, binary operators $+$ and $\cdot$ on $S$, and a unary operator $'$ on $S$ satisfying the following laws.

  (a) **Associative laws**
      – $(x + y) + z = x + (y + z)$
      – $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in S$.

  (b) **Commutative laws**
      – $x + y = y + x, \ x \cdot y = y \cdot x$ for all $x, y \in S$.

  (c) **Distributive laws**
      – $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
      – $x + (y \cdot z) = (x + y) \cdot (x + z)$ for all $x, y, z \in S$. 
Boolean Algebras

(d) Identity laws
- $x + 0 = x$, $x \cdot 1 = x$ for all $x \in S$.

(e) Complement laws
- $x + x' = 1$, $x \cdot x' = 0$ for all $x \in S$.

- If $B$ is a Boolean algebra, we write
  $B = (S, +, \cdot, ', 0, 1)$.

• Examples
  - $(\mathbb{Z}_2, \lor, \land, \neg, 0, 1)$
  - $(S, \cup, \cap, \neg, \emptyset, U)$, where $U$ is a universal set and $S = \mathcal{P}(U)$, the power set of $U$, with the operations $X + Y = X \cup Y$, $X \cdot Y = X \cap Y$, $X' = \neg X$ on $S$. 
Boolean Algebras

• Theorem
  – In a Boolean algebra, the element $x'$ of the complement laws is unique. Specifically, if $x + y = 1$ and $xy = 0$, then $y = x'$.
  – Proof.
    • $y = y1$
      $= y(x + x')$
      $= yx + yx'$
      $= xy + yx'$
      $= 0 + yx'$
      $= xx' + yx'$
      $= x'x + x'y$
      $= x'(x + y)$
      $= x'1$
      $= x'$
Boolean Algebras

• Definition
  – In a Boolean algebra, we call the element $x'$ the complement of $x$. 
Boolean Algebras

• Theorem

  – Let $B = (S, +, \cdot, ', 0, 1)$ be a Boolean algebra. The following properties hold.
    (a) Idempotent laws
        $- x + x = x, xx = x$ for all $x \in S$.
    (b) Bound laws
        $- x + 1 = 1, x0 = 0$ for all $x \in S$.
    (c) Absorption laws
        $- x + xy = x, x(x + y) = x$ for all $x, y \in S$.
    (d) Involution law
        $- (x')' = x$ for all $x \in S$. 
Boolean Algebras

(e) 0 and 1 laws
   – 0′ = 1
   – 1′ = 0

(f) De Morgan’s laws for Boolean algebras
   – (x + y)′ = x′y′ for all x, y ∈ S
   – (xy)′ = x′ + y′ for all x, y ∈ S.
Boolean Algebras

• Definition
  – The **dual** of a statement involving Boolean expressions is obtained by replacing 0 by 1, 1 by 0, + by \( \cdot \), and \( \cdot \) by +.

• Example
  – Determine the dual of \((x + y)' = x'y'\).
Boolean Algebras

• Theorem
  – The dual of a theorem about Boolean algebras is also a theorem.
  – Proof.
    • Suppose that \( T \) is a theorem about Boolean algebras.
    • Then there is a proof \( P \) of \( T \) involving only the definitions of a Boolean algebra.
    • Let \( P \) be the sequence of statements obtained by replacing every statement in \( P \) by its dual.
    • Then \( P \) is a proof of the dual of \( T \).
Boolean Algebras

• Example
  – The dual of
    $(x + x) = x$
  is
    $xx = x$.
  – Recall the proof:
    • $x = x + 0$
      = $x + (xx')$
      = $(x + x)(x + x')$
      = $(x + x)1$
      = $x + x$
  – Now the proof:
    • $x = x1$
      = $x(x + x')$
      = $xx + xx'$
      = $xx + 0$
      = $xx$. 
Summary

• Combinatorial Circuits
• Properties of Combinatorial Circuits
• Boolean Algebras
  • Boolean Functions and Synthesis of Circuits
• Applications