Discrete Mathematics

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Today's Topics

Combinatorial Circuits

Properties of Combinatorial Circuits

Boolean Algebras

Boolean Functions and Synthesis of Circuits

Applications

BOOLEAN ALGEBRAS AND COMBINATORIAL CIRCUITS

Combinatorial Circuits

- Definition
  - An AND gate receives inputs x_1 and x_2, where x_1 and x_2 are bits, and produces output denoted $x_1 \wedge x_2$, where $x_1 \wedge x_2 = 1$ if $x_1 = 1$ and $x_2 = 1$, 0 otherwise.
  - An OR gate receives inputs x_1 and x_2, where x_1 and x_2 are bits, and produces output denoted $x_1 \vee x_2$, where $x_1 \vee x_2 = 1$ if $x_1 = 1$ or $x_2 = 1$, 0 otherwise.
  - A NOT gate (or inverter) receive input x, where x is a bit, and produces output denoted $\sim x$, where $\sim x = 1$ if $x = 1$, 0 if $x = 1$.

Combinatorial Circuits

- Note
  - The logic table of a combinatorial circuit lists all possible inputs together with the resulting outputs.
- Examples
  - logic tables for the basic AND, OR, and NOT circuits
  - the combinatorial circuit in Figure 11.1.4
  - the non-combinatorial circuit in Figure 11.1.6
  - the interconnected combinatorial circuit in Figure 11.1.7

Combinatorial Circuits

- Definition
  - Boolean expressions in the symbols $x_1, \ldots, x_n$ are defined recursively as follows.
    - 0, 1, $x_1, \ldots, x_n$ are Boolean expressions.
    - If X_1 and X_2 are Boolean expressions, then (a) (X_1), (b) $\sim X_1$, (c) $X_1 \vee X_2$, (d) $X_1 \wedge X_2$ are Boolean expressions.
  - If X is a Boolean expression in the symbols $x_1, \ldots, x_n$, we sometimes write $X = X(x_1, \ldots, x_n)$.
  - Either symbol x or $\sim x$ is called a literal.

Combinatorial Circuits

- Examples
  - Computing the value of a Boolean expression
  - Finding the combinatorial circuit corresponding to the Boolean expression $(x_1 \wedge (\sim x_2 \vee x_3)) \vee x_2$

Properties of Combinatorial Circuits

- Theorem
  - If $\wedge$, $\vee$, and ~ are as in the previous definitions, then the following properties hold, where $Z_2 = \{0, 1\}$.

    (a) Associative laws
      - $(a \vee b) \vee c = a \vee (b \vee c)$
      - $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ for all a, b, c $\in Z_2$.

    (b) Commutative laws
      - $a \vee b = b \vee a$
      - $a \wedge b = b \wedge a$, for all a, b $\in Z_2$.

    (c) Distributive laws
      - $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
      - $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all a, b, c $\in Z_2$.

Properties of Combinatorial Circuits

(d) Identity laws

– $a \vee 0 = a$

– $a \wedge 1 = a$ for all $a \in Z_2$.

(e) Complement laws

– $a \vee \sim a = 1$

– $a \wedge \sim a = 0$ for all $a \in Z_2$.

Properties of Combinatorial Circuits

- Definition
  - Let $X_1 = X_1(x_1, \ldots, x_n)$ and $X_2 = X_2(x_1, \ldots, x_n)$ be Boolean expressions.
  - We define X_1 to be equal to X_2 and write $X_1 = X_2$ if $X_1(a_1,\ldots,a_n) = X_2(a_1,\ldots,a_n)$ for all $a_i \in Z_2$.
- Example
  - Show that $\sim(x \vee y) = \sim x \wedge \sim y$.

Properties of Combinatorial Circuits

- Definition
  - We say that two combinatorial circuits, each having inputs $x_1, \ldots, x_n$ and a single output, are equivalent if, whenever the circuits receive the same inputs, they produce the same outputs.
- Example
  - Equivalence of the combinatorial circuits of Figures 11.2.4 and 11.2.5

Properties of Combinatorial Circuits

- Theorem
  - Let C_1 and C_2 be combinatorial circuits represented, respectively, by the Boolean expressions $X_1 = X_1(x_1,\ldots,x_n)$ and $X_2 = X_2(x_1,\ldots,x_n)$.
  - Then C_1 and C_2 are equivalent if and only if $X_1 = X_2$.

Boolean Algebras

- Definition
  - A Boolean algebra B consists of a set S containing distinct elements 0 and 1, binary operators $+$ and $\cdot$ on S, and a unary operator $'$ on S satisfying the following laws.

    (a) Associative laws
      - $(x + y) + z = x + (y + z)$
      - $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in S$.

    (b) Commutative laws
      - $x + y = y + x$, $x \cdot y = y \cdot x$ for all $x, y \in S$.

    (c) Distributive laws
      - $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
      - $x + (y \cdot z) = (x + y) \cdot (x + z)$ for all $x, y, z \in S$.

Boolean Algebras

(d) Identity laws

– $x + 0 = x$, $x \cdot 1 = x$ for all $x \in S$.

(e) Complement laws

– $x + x' = 1$, $x \cdot x' = 0$ for all $x \in S$.

– If B is a Boolean algebra, we write

$B = (S, +, \cdot, ', 0, 1)$.

- Examples
  - $(Z_2, \vee, \wedge, \sim, 0, 1)$
  - $(S, \cup, \cap, \sim, \varnothing, U)$, where U is a universal set and S = $P(U)$, the power set of U, with the operations $X + Y = X \cup Y$, $X \cdot Y = X \cap Y$, $X' = \sim X$ on S.

Boolean Algebras

- Theorem
  - In a Boolean algebra, the element x' of the complement laws is unique. Specifically, if $x + y = 1$ and $xy = 0$, then $y = x'$.
  - Proof.
    - $y = y1$

      $= y(x + x')$

      $= yx + yx'$

      $= xy + yx'$

      $= 0 + yx'$

      $= xx' + yx'$

      $= x'x + x'y$

      $= x'(x + y)$

      $= x'1$

      $= x'$

Boolean Algebras

- Definition
  - In a Boolean algebra, we call the element x' the complement of x.

Boolean Algebras

- Theorem
  - Let $B = (S, +, \cdot, ', 0, 1)$ be a Boolean algebra. The following properties hold.

    (a) Idempotent laws
    - $x + x = x$, $xx = x$ for all $x \in S$.

    (b) Bound laws
    - $x + 1 = 1$, $x0 = 0$ for all $x \in S$.

    (c) Absorption laws
    - $x + xy = x$, $x(x + y) = x$ for all $x, y \in S$.

    (d) Involution law
    - $(x')' = x$ for all $x \in S$.

Boolean Algebras

(e) 0 and 1 laws

- $0' = 1$
- $1' = 0$

(f) De Morgan's laws for Boolean algebras

- $(x + y)' = x'y'$ for all $x, y \in S$
- $(xy)' = x' + y'$ for all $x, y \in S$.

Boolean Algebras

- Definition
  - The dual of a statement involving Boolean expressions is obtained by replacing 0 by 1, 1 by 0, $+$ by $\cdot$, and $\cdot$ by $+$.
- Example
  - Determine the dual of $(x + y)' = x'y'$.

Boolean Algebras

- Theorem
  - The dual of a theorem about Boolean algebras is also a theorem.
  - Proof.
    - Suppose that T is a theorem about Boolean algebras.
    - Then there is a proof P of T involving only the definitions of a Boolean algebra.
    - Let P' be the sequence of statements obtained by replacing every statement in P by its dual.
    - Then P' is a proof of the dual of T.

Boolean Algebras

- Example
  - The dual of

    $(x + x) = x$

    is

    $xx = x.$
  - Recall the proof:
    - $x = x + 0$

      $= x + (xx')$

      $= (x + x)(x + x')$

      $= (x + x)1$

      $= x + x$
  - Now the proof:
    - $x = x1$

      $= x(x + x')$

      $= xx + xx'$

      $= xx + 0$

      $= xx.$

Summary

- Combinatorial Circuits
- Properties of Combinatorial Circuits
- Boolean Algebras
- Boolean Functions and Synthesis of Circuits
- Applications